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ABSTRACT

This fourth yearbook on research on mathematics teaching in Finland contains five articles, four in English and the fifth in German. "Mathematics as a Cognitive Science" (C. Letteri) examines cognitive aspects related to teaching and learning mathematics, with the focus on the different levels of comprehension and an analysis of basic cognitive and metacognitive skills. "A Project for the Development of Students' Cognitive Processes in the 7th Grade of the Comprehensive School" (P. Malinen) describes experimental teaching with low-achieving students in order to develop analytical skills, use of mathematical strategies, and metacognitive thinking. "A Model for Teaching Problem Solving in Mathematics" (E. Pehkonen) considers prerequisites for problem solving and sketches a model for teaching problem solving which incorporates the solving of separate problems as well as problem solving as a teaching method. "Identification, Comparison and Justification of Central Research Orientations in Didactic Research of Mathematics" (T. Keranto) aims to initiate discussion on central research orientations in didactic research as well as the reason why specific studies are needed. The final article, "Schuelerschwierigkeiten in der Algebra" (G. Loercher), focuses on difficulties students have learning algebra and describes the development of diagnostic tests. (MNS)

Pekka Kupari (Ed.)

MATHEMATICS EDUCATION RESEARCH IN FINLAND

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MATHEMATICS EDUCATION RESEARCH IN FINLAND YEARBOOK 1986

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PREFACE

The Finnish Association of Mathematics and Science Research publishes now its fourth yearbook in the field of research on mathematics teaching. During the last two years attempts have been made also to publish a science yearbook, but a sufficient amount of articles has not been offered for publication. At the moment it seems fairly certain that the first science yearbook will come out in 1988.

Some revival and increase in the research on mathematics and science teaching has thus taken place. This was corroborated by the Didactical Seminar of Mathematics and Science organized at the University of Jyväskylä in Sebtember 1987. For the first time a joint seminar was arranged for the researchers and teachers of mathematics and science. In two days the research of mathematics, science and computer science teaching was widely covered and lively discussions followed each topic. On the other hand it should be kept in mind that studies and reports made in the field in the Finnish universities constitute only 4% of the yearly educational research carried out in Finland.

In view of the significance and esteem of the before mentioned subjects the appointment of the Committee on the Basic Education of Mathematics and Science in October 1987 was of great importance. In the Committee's work research is emphasized in two ways. Firstly, when the Committee will evaluate the present state of teaching in the subjects on the various school levels, the so far published research on teaching and learning serves as a central reference base.

Secondly the Committee's task is to find out what kinds of research and development activities are needed to promote education in the field. This could then initiate the first research program in the field.

As did the earlier yearbooks this one also contains several topics. Perhaps the most exploited topic is the activation and development of students' cognitive processes in mathematics. We regret that due to financial difficulties the yearbook comes out much later than originally planned.

Pekka Kupari

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MATHEMATICS AS A COGNITIVE SCIENCE

Charles A. Letteri

ABSTRACT

This article examines the cognitive aspects related to the teaching and learning of mathematics in schools. First the nature of mathematics and different levels of comprehension are discussed. After that the article analyses the basic cognitive and metacognitive skills in the teaching and learning.

Mathematics is often seen as a science in and of itself. That is, there is an inherent science to mathematics and it needs no other science to assist in its operations. In fact so basic is the science underlying mathematics that it is also seen as the basics of the other sciences and some of the arts as well.

With the advent of quantitative approaches to research and studies in the humanities, mathematics has emerged as the basic science of understanding and research in these fields also. There is no publication or report in any field that does not present at least some of its findings in mathematical form.

Computers have, of course, added greatly to this formulation of the presentation of information and the use of the computer has geometrically increased the use of mathematics not only in the programming of the computer but also in the more intelligent usages of the computer in professional tasks. Therefore, to deny the central role of mathematics in our everyday lives would be ludicrous.

This situation then makes mathematics all the more important within the school curriculum. Without an understanding of mathematics, statistics and the mathematically based computer languages, present and future students will be hard put to succeed in any field of endeavor.

This presentation will then examine several cognitive aspects related to the teaching and learning of mathematics in schools.

First, mathematics is a sequential as opposed to a nonsequential curriculum. That is, it must be learned in an ordered sequence if proficiency is to be achieved.

Secondly, mathematics contains all levels of comprehension. That is, it starts with rules, moves on to principles (combinations of rules) and emerges at the theoretical level (hypothetical combinations of principles).

Rules are concrete in nature while principles are symbolic combinations of rules. Both can be directly manipulated and tested, whereas theory is abstract in nature and can only be examined through hypotheses generation and testing. Some of the tests themselves are theoretical in nature and do not lend themselves to direct observation such as in astronomy and other astro sciences in which logic, coupled with deductive and inductive reasoning are utilized to propose an appropriate solution. Whereas with rules and principles the solution is observable and the response, in addition to being appropriate, must be correct.

From a cognitive perspective, using a Bruner or Piagetian framework, we can see that mathematics like cognitive development follows a path from the concrete (rules) to the symbolic (principles) to the abstract (theories).

Fortunately most, if not all, mathematics curriculum is established along this developmental framework and it is introduced more or less at the appropriate level in the child's ontological development.

However, while the curriculum itself may in fact seem appropriate to the cognitive development levels of the children, the teaching and learning of mathematics is another matter. It is excellent to have a curriculum so in keeping with the cognitive abilities of the children, what we do with it is something else.

From an information processing perspective, however, only one third of the required data is ever presented to the learner.

A learning task is comprised of the content as well as the basic and metacognitive skills required to learn the content. Unfortunately in mathematics, like all other areas of the curriculum, the teaching and learning emphasis is on the content and no teaching or learning tasks are involved with the required cognitive skills.

The basic cognitive skills include the following:

1) Analyze a problem into component parts for identification
2) Focus on the unique or relevant parts identified
3) Narrow the selection of problem category using the identified parts
4) Compare the presented problem to the selected cate gory to determine the similarities and differences between the presented problem and known problems
5) Select the appropriate solution based on the comparison process
6) Test the solution selected
7) Reselect and retest if required
8) Decide on correct solution
9) Apply solution to presented problem
10) Compare results obtained to desired results
11) Re-enter program if required

The above is a simplified version of the basic cognitive skills required for obtaining the correct solution to any mathematics problem. They are in fact the underlying cognitive skills for any problem solving task.

Yet, these are never taught in schools, singly or in conjunction with the content. How then do the children solve mathematics problems without being taught or learning the basic cognitive skills involved? Simple! They memorize solutions to classes of problems. This is accomplished through the all too familiar rote drill teaching and learning process found in the mathematics classes anywhere.

In addition the metacognitive skills of concept attainment and formation are never involved in the teaching and learning tasks either.

Concept attainment is the process whereby a learner discovers (or is taught) the underlying relationships between the various symbols employed in the task. Concept attainment, like the basic cognitive skills must also be taught and practised by the learners as well as the application and transfer of this skill to the content problems. This is not done in schools and yet forms a most important component of the learning strategies of the child. Without this strategy, the child can only hope to memorize solutions to specific classes of problems and never understand the underlying rules or principles relating solutions to problems.

Concept formation, on the other hand, is the ability to create unique solution concepts on the basis of the learned rules and principles inherent in the content data. That is, the child can attempt solutions to unique and different problems based on his/her understanding of the underlying principles involved and select solutions (or modify known solution) as is required by the specific task. The learner is no longer tied to a single memorized solution, rather using the rules and principles involved in the task, they can select appropriate solutions, combine existing solutions and create a unique solution to the presented problems. This is a dramatically different approach to the teaching and learning of mathematics than is usually found in classrooms.

The basic and the metacognitive skills required for the learning and transfer of mathematics are not taught or learned in schools.

As a result student's problem solving is usually rote in nature with solutions attempted on the basis of the problem type rather than the unique factors of the problem itself. In essence they never discover the underlying rules and principles inherent in the mathematics problems themselves and as a result have great difficulty transfering rules of arithmetic to such areas as geometry, trigonometry or calculus.

Without teaching and learning of the basic cognitive skills the child can never come to understand the underlying rules and principles effecting solution selection or solution testing procedures. As a consequence the application of rotely learned solutions is the only alternative for the child.

In addition, without these cognitive skills, the child can never come to fully understand or utilize computers in solution of problems. The child will lack the skills required for the selection and manipulation of data and solutions so basic to the present world of problem solving.

If we are to have any lasting and important impact on the education of children we must start now to include as part of the curriculum those basic and metacognitive skills required for understanding, applying and transferring content area knowledge to a wide and diverse array of problems.

We must, in the future, stop producing students who have taken mathematics and science courses and start producing students who will be the mathematicians and scientists of the future.

A PRO. -T FOR THE DEVELOPMENT OF STUDENTS' COGNITIVE PROCESSES IN THE 7TH GRADE OF THE COMPREHENSIVE SCHOOL

Paavo Malinen

ABSTRACT

The writer organized experimental teaching in mathematics for low-achieving comprehensive school 7th-graders with the objective to develop analytical skill, the use of mathematical strategies and metacognitive thinking. School achievements and cognitive features did develop during the pilot study, but the development was similar also in the control group. The present article describes the pilot project and assesses possibilities to develop the cognitive features of low-achieving pupils in school teaching.

INTRODUCTION

In August 1985 the new comprehensive school legislation came into force in Finland. At the same time ability grouping in mathematics and foreign languages was gradually abolished from the upper level of the comprehensive school (grades 7-9). The old system was replaced by additional resources for dividing students into smaller teaching groups (the time resource quota system). Consequently, the teaching groups are very heterogeneous which makes it necessary to increase internal differentiation within the teaching group.

In this situation we need to know how to help low-achievers in mathematics by giving them individualized extra training. These students have usually been given remedial instruction in mathematics already in the lower grades without successful remedy to their evidently permanent learning difficulties. An alternative approach tries to develop their readinesses to handle information and to solve problems, which implies the streering of their cognitive processes. This should lead to general improvement of their studying skills. Thus it may promote their mathematics performances in the long run, although these are not directly practised in remedial teaching. In planning this sort of instruction we can resort to the theory of cognitive psychology and the principles of the teaching of problem-solving.

DEVELOPING COGNITIVE PROCESSES IN SCHOOL INSTRUCTION

Specification of the research task

In the planning of research-based teaching processes the study of students' cognitions, i.e. their acquisition and use of knowledge, has become an important target. To parallel prior studies of ability factors, student personalities have been investigated to identify cognitive styles, which have been assembled as a cognitive profile. Cognitive style consists of basic ways of information processing, such as the skill to analyze and to concentrate on the search for essential things. When a student deals with information, he constructs cognitive processes for himself which reflect his cognitive styles. In teaching situations cognitive processes are directed by a situation - appropriate strategy which is a form of planning tied to the situation. In case a student in a problem-solving situation plans his procedure in advance, he already has a conscious strategy for the whole process. The evaluation of these strategies is often called metacognitive knowledge.

Above some of the concepts used in the study of cognitions have been described briefly. The definition and application of the concepts varies in the literature depending on the research theme and research tradition. In this paper the above-mentioned concepts are used roughly in the same way as they have been presented in general introductions to this research area, such as Leino & Leino (1982), Kirby (Ed.) (1984), Letteri (1985) and Chipman & Segal (1985). The basic concept is the cognitive process (or information processing) which is manifested in observations and tests as cognitive style, strategy or metacognitive knowledge. Of these concepts cognitive style is a more explicit concept than strategy and styles are often interpreted as being the background of strategies. Some investigations speak of strategic knowledge instead of metacognitive knowledge. In mathematics, however, it is natural to separate the mastery of computation rules (strategy) and the understanding of the problem-solving procedure (metacognition).

We shall limit this examination to situations where students study mathematics. In the same connection the aim is to develop the cognitive processes of low-achieving students. We are setting the following goals for this instruction:

1) Students' cognitive styles show development in certain traits.
2) Students adopt strategies which are suitable for studying mathematics.
3) Students acquire the habit of assessing the appropriateness of various strategies, in other words, they develop their metacognition.

In fact, these goals are quite universally presented as higher cognitive objectives to be pursued when making improvements in instruction which aims at rote learning (Chipman & Segal 1985). Cognitive objectives are not yet included in this goal-setting for the instruction, instead, the aim is to develop student personality, specifically as regards the command of processing. Traditionally, material and formal objectives have been distinguished in cognitive education. The Report of the Comprehensive School Curriculum Committee (1970) gives a careful description of formal goals and their connections with information processing and problem-solving. The development of cognitive processes is thus an important part of the cognitive education of the comprehensive school. For the present, there are very few pedagogical investigations in this domain in Finland to support teaching work. However, some basic research already exists (e.g. Leino 1985), and on the basis of stimuli from the USA, it is possible to carry out a project aimed at the development of instruction even in Finland.

In connection with this project a preliminary analysis of the following problems will be made:

1) Is it possible to construct teaching programs which will promote the achievement of the above-mentioned goals?
2) Is it possible to help students with learning difficulties in mathematics through the development of cognitive processes?

On the possibilities of developing cognitive processes

Is it then possible to develop a student's cognitive styles, adoption of strategy and metacognitions? Extensive summaries of this issue have already been presented, such as Kirby (Ed.) (1984), Segal et al. (Eds.) (1985) and Chipman et al. (Eds.) (1985). According to these, the development of cognitive style is possible on certain conditions (e.g. Meichenbaum 1985). Strategies, on the other hand, are more clearly situationally bound, but possible to adopt, although rational generalization of their application is already difficult. Evaluation of strategies (metacognitive knowledge) is therefore confined to a certain area and bound to the topic, and thus very difficult to teach in a general way (e.g. Lawson 1984).

In his definition of cognitive styles Letteri (1980) has arrived at seven dimensions whereby he defines the cognitive profile of a person. Those dimensions are usually presented in the following form:

1) cognitive complexity vs. simplicity
2) leveling - sharpening
3) tolerant - intolerant
4) analytic - global
5) broad - narrow categorization
6) focus - nonfocus
7) reflective - impulsive

Especially in the past the analytic - global dimension was referred to as field dependence. The dimensions are presented as bipolar, thus emphasizing the unique character of each end without wanting to attach any value-judgements to it. It has been found however, that the other tail of these cognitive styles is connected with good school success and the opposite tail with poor school success. This way it is possible to construct the cognitive profile of a high-achieving student (Level 1) and the profile of a low-achieving student (Level 3) (Letteri 1980 and 1985). This connection with school success has been verified also by Finnish studies, although the correlation cannot, by any means, be regarded as a straightforward phenomenon (Leino & Leino 1982 and Ristolainen & Viilo 1985). This way the different ends of a cognitive style have been given a certain order of precedence from the viewpoint of the school's educational goals. It is then appropriate e.g. to speak of a lack of analytical ability instead of globality.

It is not, however, self-evident, that we should aim at making everybody strictly analytical or reflective, since their fundamental personality structures are different. We can be satisfied, if we can break the chain of failure in the study of school mathematics by providing basic skills of information processing to those students who need them. Most students apparently acquire these skills in a normal, stimulating environment, but the least successful ones operate by the trial and error method still in the 7th grade. According to an investigation carried out in Finland, it seems that the cognitive profile remains comparatively unchanged during normal school teaching. Whan the cognitive processing of the same students was compared in the 8th and the 9th grades of the comprehensive school, only their analyticalness and reflectiveness had somewhat increased (Ristolainen & Viilo 1985). Therefore, by improving the cognitive profile, we seek to increase students' skills for information processing. According to Kirby (1984) this may promote the development of achievements connected with aptitude as well as school achievements.

Problem-solving skill is the term often used for the command of strategies which guide the cognitive processes. The analysis of the way students deal with problems they encounter in their school work has resulted in the formation of descriptive systems of heuristic strategies (e.g. Zimmermann 1983). In the planning of teaching along these lines the theoretical background is provided by the research into cognitive activity conducted by Piaget, Galperin etc. This way it is possible to describe students' performances by means of pedagogical concepts. Kretschmer (1983, 240-251) has differentiated such basic functions as assimilation through examples and advance planning of the solution stages, which are typical learning strategies. According to his research heuristic problem-solving improves when related basic functions are taught. It is thus only a question of speaking about the same things by using different names. The study of problem-solving cognition, however, starts from strategies which are directly applicable to school instruction. We utilize them in the planning of guidance, but it is nevertheless the basic research connected with cognitions which forms our theoretical starting point. When problem-solving strategies are taught as heuristic processes, it means that the stage of improving simple cognitive traits has already been passed which may initially be important especially to low-achieving students.

REALIZATION OF THE DEVELOPMENT PROJECT

The research design

There is a great amount of information available on the study of cognitive processes, but the application of such results to practical teaching has been rather insignificant so far. By the practice of cognitive styles noticeable results can be obtained in the first 6 grades, but in higher grades, in particular, it is necessary to develop suitable pedagogical situations connected with problem-solving. It is then, at the latest, that the differing capabilities of students and their variable attitudes must be taken into account.

The present writer conducted a development project of cognitive processes during the 1985-86 school year in the 7th grade of the Palokka comprehensive school. The school is a large suburban school where the number of students in the 7th grade was 132 at the beginning of the 1985 autumn term. All students in the district attend this school and no ability grouping exists in the classrooms. The teaching groups were thus of a very heterogeneous structure. Due to the time resource system the size of the mathematics teaching groups was 15-20 students and the total number of teaching groups was 8. The instruction in all groups proceeded along the same lines and according to the same textbook.

The researcher organized individualized guidance in this school once a week during mathematics lessons with the aim of realizing those objectives which were presented in the second chapter: to develop the student's cognitive style, to teach suitable strategies and to increase metacognitive knowledge. The research was started in September 1985, when all the students in the 7th grade were tested for the measurement of two cognitive styles (see Chapter 2.2). The students also took a common mathematics achievement test for the measurement of their starting level. Ten students whose achievement was low in the measurement of the cognitive styles were selected for the guidance program. The control group consisted of another 10 students who had shown similar weaknesses in the measurement of the cognitive styles. Furthermore, both these groups were clearly below average in their school achievement according to the diagnostic tests conducted at the beginning of the school year.

The researh design was a rather freely conducted field experiment in which the development of the experimental group was compared to the control group and to the whole group of students. Since the participation of the experimentals in the individual guidance session was taken from their mathematics lessons, councelling was only given for 20 minutes (half a lesson) a week, and during the session that day's mathematics tasks were also partly dealt with. Participation in the study was therefore similar to attending remedial teaching. It was found convenient and motivating sometimes to have two students attending the guidance at the same time. Guidance was given from October to March, approximately 14 times. Vacations, examination days and absence from school decreased the number of guidance sessions. In March 1986 all the

7th grade students were again tested by the same cognitive style tests as during the initial measurement. The teachers organized common mathematics exams in all teaching groups throughout the entire school year and thus the researcher had at his disposal comparable data on the students' school performances. The students' report card grades were also available. Reports on the students in the experimental group were also drafted in all guidance situations. During the school year some loss occured in the group of subjects for casual reasons (change of school, absence from testings etc.), and therefore complete data were obtained on 107 students. At the start 2 students dropped out of the experimental group, because they wanted to attend regular teaching all the time, but these students were replaced by another two students of same achievement level.

Testing of cognitive styles

In this study all the students were tested for two cognitive styles, namely cognitive complexity and analyticalness. Group tests adapted to Finnish conditions have been developed (Aimo & Viilo 1984) which have been tried out in the upper grades of the comprehensive school. Only the positive dimension of each test is used, since our aim is to develop the students' skills in cognitive complexity and analyticalness.

Analyticalness was measured by Witkin's test of "the embedded figures", in which the subject tries to distinguish a given figure from a disturbing background. After the preliminary stage the test presents two 5-minute-performance sequences and the final score is obtained by combining the number of correct items in both sequences. Cognitive complexity was measured by a task in which the subject named five familiar persons and then mentioned a common characteristic and its opposite for each person. Comparisons were made between eight couples and the score is obtained by adding up the number of different concepts used by students. The test thus focuses on the measurement of conceptual richness in language.

The measurement of analytical-mindedness has proved quite reliable. Reliability coefficients have been high and the stability coefficient obtained after a period of one year was 0.79 (Aimo & Viilo 1984 and Ristolainen & Viilo 1985). In the present study the reliability coefficient for the different parts obtained by the corrected split-half method was 0.87 (initial measurement) and 0.86 (final measurement). The stability coefficient between the initial and the final measurements was 0.77 (n=107). The stability of cognitive complexity according to the former investigation, after an interval of one year, was only -0.10 (Ristolainen & Viilo 1985). Randomness in the number of used concepts has been evident. In the present study the correlation coefficient between initial and final measurements was 0.40, indicating that ramdomness is a very general feature in the measurement of cognitive complexity.

The instruction program

For the experimental group students (n=10) 20 minute guidance sessions were arranged approximately 14 times in a separate group instruction room. The present writer professor Paavo Malinen taught one half of the student group and the other half was taught by ME Arto Haaraniemi and Heikki Tyrväinen. The teaching programs were prepared in advance by the team. At first Dr Charles Letteri's instructional programs were used. He had presented these programs in 1984 and 1985 when he visited Jyväskylä. These programs are, however, only partly suitable for the instruction of the 7th grade, and therefore the present writer constructed a number of additional programs on the basis of various sources. Some of the training programs were based on the tests presented by McGinty et al. (1985). These tests, according to the authors, involved deductive and analytical thinking.

When determining the level of cognitive processes the instructor, together with the student, first examined the topic under study in the mathematics book and then tested the student's skill in demonstrating the content of the studied subject matter, the discussed topic as a unit, and the progression of his thinking process. In the construction of this test inspiration had been found in the articles of Peterson et al. (1982 and 1984). It was also from the same basis that a training program emerged, in which the student answered a questionnaire describing the way he or she studies mathematics. This task was presented to the students twice with approximately a month's interval. In both training programs attention was already paid to the students' skills to assess their own cognitive processes.

The instructors agreed on common ways of guiding the students in the teaching situations. The goal was to develop cognitive styles, mathematical performance strategies and metacognitions related to the study of mathematics. In addition to these goals objectives which are typical to problem-solving situations were also aimed at, such as:

- giving an oral account of one's own thoughts and the rules which were applied
- focusing one's thoughts on essential information
- comparing rules connected with different mathematical procedures, and their use.

The teaching program involved situations related to the use of heuristic methods only in some training programs and thus the main emphasis was on the development of thinking to help students in adequate application of previously acquired knowledge. The 10 exercises which formed the core of the instructional program are available as a separate copy from the present writer.

The practice commenced with the improvement of analyticalness and attentiveness which focused on the examination of certain features (e.g. computation rules) by eliminating irrelevant information. Parallel to the practice programs current topics in the mathematics book were also discussed, then concentrating on concept formation. An essential part of the practice on these occasions consisted of the students' own talk and clarification of ideas. Somewhat later comparative analysis and construction of rules were also included. This was realized partly through the practice programs, partly through textbook tasks. The applied guidance procedure was shaped after the presentation of Letteri (1985, Chapter 6) and it corresponds to a universal pedagogical conception of the progression of teaching from simple issues to general ones.

Some parts of the practice programs were easy, some of them were difficult for these low-achieving students. What was considered essential was to elicit a discussion on the solutions. Most students in the experimental group performed the usual mathematics tasks even in the guidance situations without being able to say anything about them. When problems of a new type emerged in the practice programs, the explication of rules connected with them was operationally something new. Cognitive processes were thus stirred up in course of the practice programs, but it is not possible afterwards to examine the effects of the teaching very comprehensively.

RESEARCH RESULTS

Development of cognitive processes

Information concerning the development of cognitive processes was obtained from all the students by means of initial and final testings of two cognitive features (cognitive complexity and analyticalness). In addition, information concerning the level of their cognitive processes was also obtained from the experimental group students (practice program 7) as well as concerning the accounts of their study habits (practice program 10). On the other guidance situations merely observations were made, which have not been systematically assembled.

The development of cognitive complexity in the different groups is presented in Table 1. The experimental group (n=10) and the control group (n=10) were at first below average. In the retest the control group's performances surpassed those of the experimental group. The difference did not, however, turn out to be significant. The test has been found to have a weak reliability, and thus the development in all groups is partly caused by randomness, and partly by familiarity with the testing situation.

The development of analyticalness in the different groups is presented in Table 1. The experimental group and the control group were first below average also in this respect, but the development of the experimental group now exceeds that of the control group. Both groups have come closer to the average level. Apparently, in the initial testing, these students had experienced difficulty in adjusting to the measurement situation and were now concentrating better. They also had room for improvement, whereas the best students had reached the ceiling and therefore could not demonstrate further development. The results of the experimental group may also indicate the effect of training which had included tasks related to analyticalness. The training did not, however, have a statistically significant effect on the growth of analyticalness.

TABLE 1. The results of the initial and final measurements of cognitive features.

	Cognitive complexity				Analyticalness			
	Initial testing		Final testing		Initial testing		Final testing	
	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
Experimental group (n=10)	8.4	2.8	9.4	3.2	4.2	2.9	10.4	2.7
Control group (n=10)	8.0	2.8	10.0	2.9	3.9	2.6	9.0	5.1
Whole group (n=107)	9.6	2.8	10.6	3.3	9.6	5.0	12.9	4.2

The instructors estimated the level of cognitive processes by means of practice program number 7 in January 1986. The students could give a clear account of what problems they had been able to solve, but they could not name the arithmetic processes nor explain the applied computation rules, unless the teacher gave them some clues. Almost all students found it impossible to give an account of the composition of a subject matter unit and its connections with things that had been learned earlier. This situation clearly revealed that problems were solved by means of concrete cues without any consideration to the background of the process. On the basis of the instructors' notes no objective measure could be formed for the level of cognitive processes and no retest was conducted after the experiment. Thus no other significant results were obtained in addition to the above-mentioned finding concerning the weakness of conceptual thinking.

Practice program number 10 consisted of 12 questions asking the students to describe how they study mathematics. The students estimated their thinking processes by the following scale: "almost always, often, sometimes, not usually, I don't know". This cannot be considered as an interval scale, but since the last alternative was very seldom used, students' ratings were transferred into a numerical scale 1 - 5; the sums of scores and averages were computed from these. The questions were repeated at about one month's interval. The item-specific test-retest correlations varied between 0.00 - 0.97. High reliability was found in items 2, 5, 6, 11 and 12 which were concerned with executing processes or direct control of cognitive processes. Question number 2, for example, was: "Did you apply some arithmetical rule in your head while solving the problem?" Weak reliability (0.00 ... 0.20) was found in items 3, 4, 8 and 9 which concerned comprehensive information about cognitive processes, which can be regarded as metacognitive knowledge. Let us take question 4 as an example: "In doing the essay tasks, did you consider the causal relationships of the concepts involved?" The mean of nearly all the answers varied between sometimes - often, which shows that, according to the students' ratings, even these difficult things had been given some thought. Notably high estimates were given as to the following of execution principles of the problems and understanding the contextual unit (items 8 and 9). It is the student's 'ideal self' which has been manifested here, for these are the very things that are not mastered in practice. The students' estimates of their own studying style did not essentially change during the month's interval, although the replies were very random due to the small variation.

The students' sums of scores were computed from the 10 items of the practice program, although correlations between the items were weak (-0.22 ... 0.36). It seemed, however, that the overall planning of studying was relatively stable, for the correlation coefficient of the sum of scores between the two points of measurement was 0.89. This group of students therefore exhibits constant variance, although it is evenly low in its school success. Students' assessments concerning the advancement of their own studies were more reliable than teachers' estimates, according to the study of Peterson et al. (Peterson et al. 1984). The same result was obtained in this study, although the students' assessments can be regarded as beautiful wishes rather than real situational estimates.

Development of mathematical performances

At the upper level of Palokka comprehensive school common examinations were organized during the school-year 1985-86 in the 7th grade for all parallel groups. This way comparable measurements of school achievements were obtained: the initial level at the start of the school year, plus 6 exams. We also had access to the mathematics grades in the students' report cards given in December and May. Table 2. shows the results of school achievements. The scoring scale ranges from 4 to 10 in exams and report card grades. The experimental group and the control group have made same progress, for the level of school achievements has risen in both. The same development can be seen also in the whole group, although not as clearly. The experimental group and the control group have thus approached the average level. In both groups 4 students have improved their grades from Christmas to spring, while the corresponding increase in the rest of the group (n = 87) is 15 grades. None of the 7th grade students received the failing grade (4) in their spring report card.

TABLE 2. School achievement results

Measurements of school achievements	Experimental group (n=10)		Control group (n=10)		Whole group (n=107)	
	$\bar{x}$	s	$\bar{x}$	s	$\bar{x}$	s
Starting level	5.8	0.7	6.0	0.7	6.7	1.3
Test 30.9.	6.4	1.1	6.9	1.0	8.0	1.5
Test 5.11.	6.5	1.2	6.5	1.3	7.7	1.5
Test 10.12.	5.8	1.0	6.1	1.3	7.1	1.7
Christmas report card	6.2	0.6	6.5	1.2	7.5	1.4
Test 17.2.	6.8	0.9	6.9	1.3	7.9	1.5
Test 15.4.	6.6	1.6	6.6	1.7	7.7	1.7
Test 19.5.	6.3	0.8	6.8	1.3	7.7	1.4
Spring report card	6.6	0.8	6.9	1.1	7.7	1.4

According to the teachers' assessments students in the experimental group actually strengthened their performances during the winter, but no systematic development could be detected. These students' working habits were partly casual, since at times some even ignored their homework. One student from the experimental group was transferred to a special class in the autumn. Also some others had general problems related to school and their overall school achievements were below average. For the low achieving students outside the experimental group ordinary remedial teaching in mathematics was provided, which partly explains, why there were no failing grades. The guidance that was given to the experimental group students had the same effect on school success as normal remedial teaching.

A central topic in the 7th grade mathematics course is the reduction of algebraic expressions, such as arithmetic operations on monomials and on fraction classes. Other topics include computing the areas of figures and the volume of bodies, solving equations, etc. The course is relatively demanding, considering that the whole age group is involved, with the exception of those few students who have been transferred to special education. All the students improved their mathematics performances in these areas. The tests mainly measured computing skills as well as the skill to apply given arithmetic rules. The essential thing was to use convergent thinking when choosing the right computation methods and to arrive at the right solution. Since the instruction moved comparatively fast to new topics and algebraic expressions were handled by means of abstract rules, to be successful students had to be able to apply high-level cognitive processes and familiar strategies. These performances have been little analysed from the viewpoint of cognitive styles or processes. Their connections with mathematical abilities have been investigated and thereby it can be stated that it is possible to teach problem-solving strategies in mathematics in order to improve mathematics performances (e.g. Briars 1983). The rather small amount of guidance which was given to the experimental group in this matter did not seem to produce clear results.

Experiences yielded by teaching situations

The reports on the teaching of the experimental group gave a short description of the progression of the instruction, students' reactions etc. Although the advance plan prescribed that the same exercises should be taught to all students, it was necessary to change the program according to the interests and capacities of the students. Most students were capable of concentrating on the 20 minute guidance period for only part of the time. Some tried to solve the presented problems by random guesses, while others gave them long consideration without really understanding the idea. Teacher's help was often needed in the examination of figures, and thus the students had to solve only narrow problems. In all situations the student was encouraged to explain actively his thoughts and to ask for help in his problems. Student's own problems did emerge at times and were discussed. Most of the time, however, was used in teacher's presentation and questioning as well as students' thinking.

Some of the students thought that it was interesting to deal with the problems not included in their school mathematics, but others preferred the treatment of the same topics as in the lessons, in order to get a great number of correctly solved problems in their exercise books. Even the low-achieving students thus had a need to produce correct answers, but the problem-solving process was connected to concrete stimuli (such as + and - signs). Many students succeeded in reporting on the phases of the problem, but the use of concepts in their speech was nonexistent. Most were reluctant to say anything, and it seemed as if they only saw a limited set of stimuli in the situation to which they randomly sought to find a possibly correct response. Occassionally, at least, broader entities and computational principles were understood.

The guidance situations seldom involved demanding development of cognitive styles (mainly analyticalness). The short guidance sessions hardly yielded any feedback which could have been used for further development of the process. Consequently, in practice we aimed at the development of situationally bound strategies and, from time to time, asked the students to evaluate their own strategies. When the student's own thinking moved in the S-R type connections, it was difficult to interest him in more general topics. The tutoring thus turned into presentation of simple arithmetic problems in order to preserve the motivation towards this study program. Evidently the students found it difficult to leave the rest of their student group for the 'professor's remedial instruction', where they had to think, explain and answer questions all the time. Consequently, it was not very often that the guidance reached a very high level of developing mental processes.

ON THE DEVELOPMENT POSSIBILITIES OF MATHEMATICS INSTRUCTION

If compared to many other research studies aiming at the development of cognitive processes, this project had the following special characteristics:

1) Guidance was carried out parallel to regular school instruction, as remedial teaching. The participating students had a weak motivation

in studying some things external to mathematics, while they were expected to learn the normal course. Therefore in the guidance we sought to combine the development of cognitive styles and strategies with the study of algebra, which in the limited time span was a difficult solution.

2) The students in question had already studied mathematics for 6 years and had been given extra guidance when necessary. Nevertheless, their school success was poor and they were used to thinking that they are failures in mathematics.
3) In the Finnish comprehensive school we try to help the whole age group to learn mathematics for 9 years. Therefore it is still necessary to assist all students in the 7th school year, even though they would have no desire to learn themselves. In many countries algebra and geometry are no longer taught to everyone at this stage (lower secondary education, age 14 - 16).

For the above reasons we had to give more attention to the examination of students' motivation than many other reports. Earlier studies have found that the development of students' cognitive styles is affected by values, hopes and habits (Baron 1985, 385-386). Styles again influence a person's use of strategies. Thus a student may think ineffectively in the domain of mathematics unless he has attached strong hopes to it. It seemed that also the experimental group students only expected somehow to get a passing grade in mathematics which they believed they would get by applying simple computing methods. It is possible that due to this blocking attitude, it is not worthwhile to try to develop the cognitive styles of these students in connection with mathematics instruction. Highly positive results have been reported on the development of cognitive styles and strategies. Meichenbaum (1985) notes, however, that it is not often that any generalizable results have been shown. Transfer to other areas will not succeed until the process has been made more profound. This already sets great demands on the teacher's activity. Ready-made instruction packages are of small importance, since the starting point of the guidance must be the student's own thinking process. A metacognitive perspective must be achieved in the instruction and therefore the instructor needs a good goal analysis which he adapts to the student's capabilities. J. von Wright (1986) has arrived at similar conclusions. According to him cognitive psychology can help the teacher when he sees suitable opportunities for improving his teaching in constantly changing situations. Since individual factors play an important role in the learning process, it is not possible to offer clear and simple recipes to the teachers.

The experiences yielded by this project support the above findings. Short-term external guidance cannot provide very effective help. The conditions set for good instruction by Meichenbaum (1985) were poorly met by this project. The teaching situations remained too superficial in view of the development of the student's own thinking process. It would be best if normal instruction could be made to support the development of the student's thinking process. Then the treatment would not be limited merely to the development of cognitive styles, but would also include the teaching of well-motivated larger functional units. When mathematics teaching is connected to this from the start, students will accept it as a goal and value the study of mathematical strategies and metacognitive examination. Another possibility is by means of voluntary guidance to develop students' cognitive styles as a long-standing and intensive project, but the transfer to the study of mathematics may then be difficult. The results so far available to the above problems are still tentative:

1) It was found possible to construct a teaching program of an appropriate difficulty for 7th grade students. Its content is many-sided, but at the same time rather disconnected, because the program was supposed to promote the development of analyticalness, the study of basic algebra and metacognitive thinking. This teaching program provides possibilities for many kinds of further development.
2) It is difficult to say, on the basis of these experiences, whether the development of cognitive processes can be used to help students with learning difficulties in mathematics. Since the experimental group students have had constant difficulties in the study of mathematics, motivation for this type of study is also weak or occasionally good at the most. In some cases the teacher has nevertheless succeeded in getting the students to discuss their mathematical performances and to use the students' internal language. This way students' self-knowledge and goal-directedness

have increased, although it is not clearly perceptible in school achievements.

The project for the development of cognitive processes which has been described in this report leads us to examine possibilities to continue this research. The training of teachers is without doubt a central task. It is necessary to provide the teachers with concrete practice material and a great deal of guidance along their everyday work in order to activate the development of their own thinking processes. During his visit to Finland in autumn 1985 doctor Letteri made some efforts in this direction. It would appear to be more effective to start this sort of guidance on the lower level than on the upper level of the comprehensive school. It seems promising against this background to connect psychological knowledge with actual teaching work, although for the present there are no possibilities of making rapid progress in practice. Therefore the prospects are very good in view of further resea~ch and development projects.

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A MODEL FOR TEACHING PROBLEM SOLVING IN MATHEMATICS

Erkki Pehkonen

ABSTRACT

We are going to discuss some aspects of teaching problem solving. After considering the prerequisites for problem solving (a favourable atmosphere, creative activity, positive attitudes, cognitive skills), we will sketch a model for teaching problem solving. The model takes into account the solving of separate problems as well as problem solving as a teaching method.

INTRODUCTION

It is often said that the point of emphasis in school instruction should be changed from teaching contents (substance objectives) to developing thinking skills (formative objectives). This change of emphasis in school instruction will be supported both by researchers of education and by representatives of industry and economy.

According to the cognitive viewpoint of learning, the changes caused by learning can appear both quantitative and qualitative. A quantitative change takes place e.g. when the amount of knowledge is increased, whereas a qualitative change refers to such a process, where changes will occur in the structure, in the order and the meaning of memorized knowledge. When we change the point of emphasis from learning calculations to developing thinking skills, we are just trying to gain qualitative changes along with quantitative changes in school instruction. Commonly, problem solving is accepted as a method when trying to develop thinking skills.

Here the starting point will be the definition of a problem which is commonly used in the literature (see e.g. Kantowski 1980, 195): We will call a problem such a task situation where the individual is compelled to connect the acquired information in a (for him) new way, in order to do the task. If he can immediately recognize those actions needed to do the task, it will be a routine task for him. It is worth noting that the concept 'problem' is bound to time and the person.

We can now define problem solving "as the process of applying previously acquired knowledge to new and unfamiliar situations" (NCSM 1977, 20). So, the problem solving process means all the actions which appear when an individual is solving a problem.

The question why to teach problem solving in school is not discussed here, because it is dealt with in another paper (see Pehkonen 1987).

PREREQUISITES FOR TEACHING PROBLEM SOLVING

For the development of problem solving skills, it is important that the pupil has an opportunity to try, to guess, and to make mistakes alone, or in a small group without time limit (Trauerstein 1981, 126). Experience has also shown that for creative activity (as problem solving), working in groups seems to be more effective than individual work, if the group is not too large. Working in pairs is perhaps the best form, because then the exchange of ideas, which is important for creative activity, occurs but the size of the group should be as small as possible. Then as many of the pupils as possible are thinking independently.

Besides working methods, the attitudes of the teacher and the organization of instruction are also important. The teacher's role should change from model to facilitator during the teaching of problem solving within a long time span (Kantowski 1980, 198). In addition, the learning atmosphere in the class has a great influence on success in problem solving. Obstacles for creativity (and so for problem solving) are e.g. teacher's unifying actions, authoritarian performance, scornful behavior, emphazising of students' assessment, searching for security and stressing the meaning of success (Kiesswetter 1977, 4).

When classifying the difficulties met by pupils in problem solving, Moses (1982, 11) puts the basic skills at level 2. But low achievers in mathematics have usually gaps in their mathematical knowledge and also in their application skills. Therefore, one way to develop positive attitudes is to practise problem solving in the beginning with mathematically simple problems, in order to pass the difficulties at level 2.

In summary we can state that the prerequisites for teaching problem solving successfully seem to comprise of the following actions:
(1) to create a favourable atmosphere in the class,
(2) to add readiness of pupils for creative activity,
(3) to foster positive attitudes in pupils,
(4) to develop pupils' cognitive skills.
These four points could be placed under one action rule:
(5) to develop problem solving persistance in pupils.

When teaching low attainers in mathematics, it is most important to create problem solving persistance in pupils. One way might be to do it through recreational mathematics. With mathematical puzzles, problems with matches etc. the teacher can give pupils experiences of success, which are essentially important when developing problem solving persistance. More about the meaning of such mathematically simple problems can be found in Pehkonen (1986).

PROBLEM SOLVING AS A TEACHING METHOD

When teaching problem solving we should not stop here, but continue: Our goal should be to change the whole instruction into being as much problem-centered as possible. The most important thing is no longer the right answer, but rather the action that leads to the answer. This will change the point of emphasis from product to process.

If the point of emphasis in the objectives of the mathematics curriculum is changed from numeracy to the development of thinking skills, the mathematics instruction can be built into such frames that correspond to the needs of all pupils. Logical thinking and creativity are needed in all fields of life. Because the aim of mathematics teaching is now the development of mathematical thinking, the emphasis of contents to be taught can be selected accordingly. The teacher can concentrate on such topics where the development of mathematical thinking and creativity can be most easily realized.

Two articles provide examples of such teaching. Grayton H. Bedford (1984, 262) states that "any topic in the secondary school curriculum can be presented in a way that exercises thinking skills". He shows with some examples from algebra how to do that. Werner Walsch (1985) describes the reform in mathematics instruction realized in the GDR: The aim is to get out from the "learning from the master" -method to the development of problem solving thinking. The leading idea is to deal with ordinary mathematical tasks in an unordinary form, i.e. problems to be dealt with in the classroom are changed from closed to open-ended ones.

In the methodology of problem-centered teaching, the role of the teacher as a 'normal teacher' is minimized and contribution of pupils is growing. When the development of thinking skills is the objective, pupil-centered working methods are more effective than teacher-centered ones. When using group work, differentiation can be easily realized in the class. Additionally, the teacher has time to guide the pupils individually.

TEACHING HEURISTIC STRATEGIES

Heuristics can also be taught mechanically: The teacher introduces different strategies, shows with examples how they are applied in solving problems, and urges the pupils to try them systematically. Such problem solving courses are arranged amongs others in USA in many educational institutions (Boykin 1985, 88) and handbooks for such problem solving have been published (e.g. Wickelgren 1974; Krulik & al 1980).

I think that the teaching of heuristic strategies is sensible only when creative problem solving has been practised for a longer time with a rich variety of experiences. Early training of heuristics can be formed into a routine to be taught in school.

The most useful heuristic strategies are important for the teacher to know and to be aware of when to use them. Therefore, there should be a course of heuristics in the mathematics education given at the university level to new teachers.

A MODEL FOR TEACHING PROBLEM SOLVING

Based on the above-mentioned ideas, the following three-staged model for teaching problem solving in school mathematics can be built. The model takes into account both the solving of separate problems (as e.g. problems with matches) and problem solving as a teaching method.

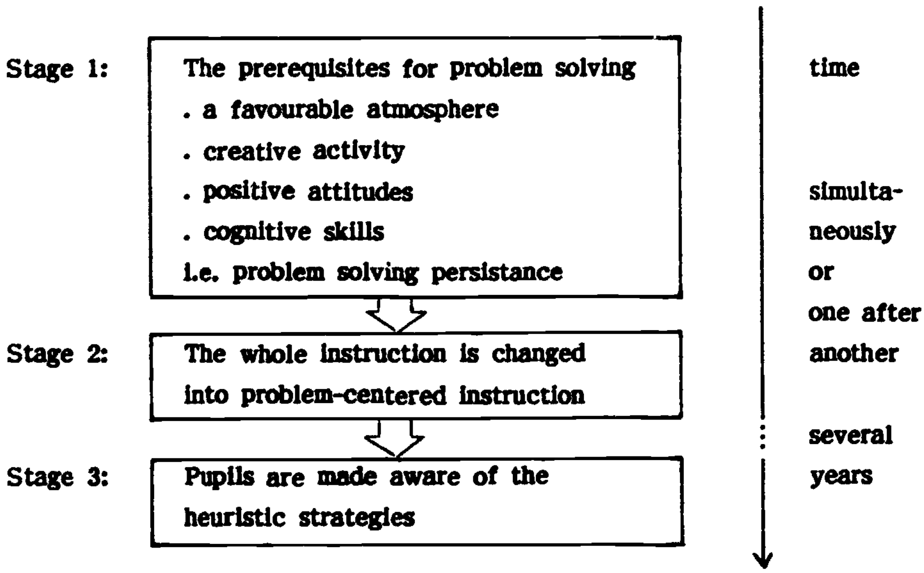

FIGURE 1. A model for teaching problem solving

The first and second stages can be carried out, in instruction, simultaneously or one after the other. But between the second and third stage pupils ought to have sufficient experiences in problem solving over several years. Thus, at stage 1 and stage 2 pupils are given a possibility through solving problems to form their own solving strategies. At stage 3 a summary of the heuristic strategies used in problem solving will be made.

Although stage 2 has the title "The whole instruction is made problem-centered", it does not mean that routine exercises do not have any place in mathematics instruction. The purpose now is to try and change also practise situations into problem solving, if possible, or at least to get some thinking exercises besides routine drills. The same idea is to be found in Wittman (1986), where he emphasizes that when speaking about reflective practise there ought to be some uniting idea behind drill exercises. Therefore, with each mechanical drill, there will be a task of the second level (thinking level).

CONCLUDING REMARKS

Teaching problem solving creatively has been discussed for years, but in practise the method of "learning from the master" has been realized. There are always teachers who teach more 'openly', but most teachers use the traditional teaching model.

Perhaps, the 'keys to change' could be found in teacher education? If we could somehow succeed in helping teachers to get rid of their formal thinking schemata and ready-made solving models, their chance to act in a completely new way and creatively would increase. Therefore, one possibility for change could be a creative and innovative mathematics teacher! So we come to the question how to foster the development of creativity in mathematics teachers. As an answer I offer the same methods as in the case of pupils: personal activity - e.g. solving simple problems, geometrical investigations (building models etc.), playing with numbers, playing and developing learning games.

Some people see computers as 'magical tools' one can do almost anything - e.g. to teach problem solving (research on artificial intelligence!). A computer can at its best develop the logical thinking of its user e.g. with games like Master Mind. But a computer can never produce creative performance.

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IDENTIFICATION, COMPARISON AND JUSTIFICATION OF CENTRAL RESEARCH ORIENTATIONS IN DIDACTIC RESEARCH OF MATHEMTICS

Taplo Keranto

ABSTRACT

The aim of the paper is to initiate discussion on central research orientations in didactic research as well as the reason why we need specific studies. An attempt is made to answer three central questions:
(a) Which are the central research orientations in the research of mathematics learning and teaching and their goals, tasks and means?
(b) How do the research findings and methods of the identified research orientations relate to each other?
(c) What possibilities do the studies oriented in each way havein attaining the theoretical and practical goals of scientific research?

As a result of this preliminary study five research orientations were identified: i.e. behavioral, structural, processual, psychometric and pragmatic orientations. Comparing the research findings in the orientations in question it was concluded that behaviorally, structurally and psychometrically orientated studies can be made more precise, complete and thorough by the aid of processually orientated research. It was also found that the rational and empirical task analysis typical in processual orientation was also needed in pragmatically orientated research when designing and realizing the efficiency of teaching programs. This led to the discussion on the justification of each research

orientation i.e. why do we need and do we need studies orientated in a certain way when striving to develop science and attain practical aims.

INTRODUCTION

The aim of this paper is to initiate discussion on central research orientations in didactic research and to find out why we need research orientated in a specific way. By research orientation I mean research that is orientated and realized in a certain way, and whose pursuers have a consistent view on what is the true nature of the research object, and about which characteristics we can get reliable information and by which means the information can be attained.

Research of mathematics learning and teaching - as educational research in general - is here understood as intentional goaldirected activity. Taking this viewpoint there is no reason to assume a priori that research orientated in a certain way would be rational and would lead to valuable and useful goals in advancing didactical research. In other words research orientated in a specific way must be grounded or justified. How should this justification be done? Primaly we need two kinds of factors: (a) value judgements concerning the goals of scientific research, (b) all knowledge that can be attained from the findings and methods of each research orientation in the study of mathematics learning and teaching. From the factors in question we can derive conditioned recommendations for action or technical norms, which have a twofold task: (a) they can justify or reject the act, in relation to which they are relevant, (b) they can motivate the researcher to do an act (or to restrain from doing it), in relation to which they are relevant (cf. Lesche 1976; Tranöy 1976).

The aforesaid raises three central questions which I shall try to answer briefly in the following (see closer Keranto 1986a,b).

(a) Which are the central research orientations in research of mathematics learning and teaching and their goals, tasks and means?
(b) How do the research findings and methods of the identified research orientations relate to each other?
(c) What possibilities do the studies orientated in each way have in attaining the theoretical and practical goals of scientific research?

This way we can start talking about the justification of the studies orientated in each way in a sufficiently detailed way and in a specific area. At first I shall consider the goals of scientific research and the knowledge interests connected with them. Secondly I shall identify central research orientations in the study of mathematics learning and teaching and their goals, tasks and means. Thirdly I shall discuss the justification of studies orientated in each way.

The value of results and knowledge attained by scientific research

Justification of research orientated in a certain way is tied to the goals of scientific research, which can roughly be classified into two main categories:
(a) Theoretical goals; attaining as systematic and truthful a view about nature, man and society as possible;
(b) Practical or instrumental goals; search for applicable knowledge in order to control, understand and emancipate nature, man and society.

If we accept information about reality as a primary goal of scientific research, there exist obvious possibilities of uniting the presented clauses of scientific research activity (see Niiniluoto 1980, 73-77). More accurately it could be of the greatest value to the scientist in his research activity to try to acquire as systematic and truthful information as possible. The term 'truthfulness' refers here to the uniting of truth and information so that among the statements concerning mathematics learning and teaching the most informative truth is the strongest and the trivial, tautological truth is the weakest (see truthfulness, Niiniluoto 1984, 44-45, 48-54; Niiniluoto 1986). In other words research of mathematics learning and teaching does not aim at any true propositions about educational reality, but to results rich in information and as near the real state of facts as possible. The aim is to construct a systematic and holistic view on the learning and teaching in different sub-areas of mathematics and about their interrelationships. The function of the attained information would be to describe and explain the facts and regularities in the sub-area of the educational reality in question. The scientific models and theories of learning in each specific mathematical sub-area serve as a vehicle in carrying out the task. Theoretical aspiration for truth for its own sake is worthy because its fulfillment contributes in analyzing and outlining the learning and teaching of mathematical sub-areas. We can talk about the theoretical interest of knowledge, which means that the theoretical aim for truth is rational and valuable, because its realization makes it possible to describe and explain the sub-area of educational reality in question. The aim of attaining a truthful view on mathematics learning and teaching and on their interrelationship can have instrumental values, too. As far as research results and methods make it feasible, beside the theoretical aim, the control and guidance of the learning process and the factors affecting it, we can in Habermas' terms talk about pursuing 'the technical interest of knowledge'. This argument is valid also in the case where the direct function of research is the prediction of learning acts, and when the aim of research is the guidance and control of learning and of the language and thinking needed in it (cf. Habermas 1975; Niiniluoto 1980, 69-73). On the other hand, if the methods and results of the research orientated in a certain way makes possible the growth of students' and teachers', 'selfunderstanding' and their mutual appreciation, its value can be justified from the viewpoint of 'the hermeneutic interest of knowledge'. If the methods and results of specifically orientated research make feasible the emancipation of teacher and students from faulty assumptions, attitudes, judgements and from insignificant and meaningless activities, its value can be justified by emancipatory interest of knowledge.

CENTRAL RESEARCH ORIENTATIONS IN MATHEMATICS LEARNING AND TEACHING

Traditionally the central dividing line in the research of mathematics learning and teaching used to be between calculative skills and mathematical understanding. This is astonishing because it would make sense to assume that both these aspects of mathematics learning and teaching were closely related. In other words it could be assumed that lacking valid conceptual knowledge about the principles behind symbolic manipulations these activities wouldn not have any real, objective significance or meaning (cf. routine and meaningful learning, Ausubel 1968; significance and meaning of activity, Leontjev 1977, see also didactic phenomenology, Feudenthal 1983). Reversely it could be assumed that the mastery of specific calculative and strategic skills is crucial for meaningful mathematics learning. Accordingly it could be assumed that research on mathematics learning and teaching would mainly concentrate on studying the relations between the aspects in question. This is not, however, the case from the historical perspective.

In fact we can identify two research orientations in the research of mathematics learning, the one - the behavioral' orientation - concentrated on studying observable calculative skills and the other - the structural orientation - on studying understanding of mathematical tasks. The origin of the behavioral orientation lies in the experimental laboratory studies of learning mechanisms (Thorndike 1913, 1922). In the orientation in question the idea of mathematics as a system of calculative rules and skills is emphasized. The main idea and ontological commitment is that mathematics learning can be reduced to S-R-linkages (see Gagne 1973, 62-66, 246-256). On the basis of content analysis the aim of behavioral orientation seems to be a description of the learning and teaching of mathematical skills in the terms of observable behavior.

The task is to guide, control and predict the mastery of these skills. As vehicles it employs learning hierarchies, the development and empirical examination of which are carried out by the aid of scaling and rehearsing studies (see learning hierarchies, Gagne 1973, 251-252; Gagne & Briggs 1979, 107-110, 147-151). The methods of the behavioral orientation could be characterized as follows:

(a) Rational analysis of the hierarchical system of learning tasks in a specific mathematical content or topic area;
(b) Empirical testing of the hypothetical hierarchy by the aid of training studies and scaling techniques;
(c) Making the hypothetical hierarchy more precise, improving it or rejecting it.

Why do learning hierarchies, if they have proved valid at all in empirical studies, exist and how and why do they function? Which subjects in mathematics learning can be developed into learning hierarchies? Is proceeding in learning hierarchy the best possible way to study mathematics from the perspective of understanding and meaningful learning? What kinds of mental operations and intellectual constructs or structures does the solution of each mathematical learning task involve and how do these operations and structures develop? Those are the questions, to which the behavioral orientation does not easily provide answers. Thus we need differently orientated studies, which pay more attention to the individual's intellectual construction and to the solution processes of mathematical problems.

The main idea of structurally orientated research reveals itself in the conception of the intellectual structures which guide the individual's operations and which cannot be changed without changing the structure of the whole system. In other words mathematics learning should be described by referring to the development of the intellectual constructs needed in solving mathematical tasks. Piaget's structuralism can be regarded as a main representative of the structural orientation (see Flavell 1963; Piaget 1970, 1971). The aim of the orientation in question could be seen as a construction of a system, which describes spontaneous development of mathematical- logical thinking. Subsequently its task would be to describe and explain mathematical-logical thinking required in mathematics learning and problem solving. As means it would use mathematical models developed for the purpose in question or those borrowed from mathematics (see 'groupings' and INRC-groups, Flavell 1963, 215-222; Inhelder & Piaget 1958). The methods of the orientation in question could be characterized as follows:

(a) Stating or formulating mathematical problems having a specific abstract mathematical construct;
(b) Data collection based on clinical interview, qualitative analysis and description and definition of developmental stages and levels;
(c) Development and perfection of mathematical models describing the intellectual constructs in each developmental stage and level.

The methods in question do entail problems. How do we, for instance, define the competence related to each developmental stage and level? Because we cannot directly observe hypothetical intellectual structures, we must infer them from the performance or reports of the interviewees. If the problems used in the evaluation of a specific developmental level or stage are not competently formulated, the conclusions made may be greatly erroneous. In fact further studies refer to the problematic character of the tasks used by Piaget (see e.g. Brainerd 1978, 1979; Klahr & Wallace 1976). Secondly we can ask, how could the results based on a clinical interview be replicated? I think that striving for truthfulness and systematic information about mathematics learning and thinking in general implies the possibility of replication. Thirdly, is it relevant to try to describe and explain the development of mathematical-logical thinking by the aid of abstract mathematical models hypothesized to underlie the solving process of each task type? In other words can the individual's intellectual development related to each mathematical learning task be described and explained competently and in an adequately detailed level by these models?

Processually orientated neo-Piagetian studies refer to the fact that the models originally developed by Piaget are extremely rough as to describing the intellectual development. In fact many Piaget's original ideas and research findings have been completed and made more accurate in processually orientated studies (see for instance Case 1980a,b; Klahr & Wallace 1976; Noelting 1980a,b; Pascual-Leone 1970; Pascual-

Leone & Smith 1969).

In the processual orientation which has been referred to before, learning is understood as changes occurring in the individual's information processing system. Attempts are made to describe these changes in terms of the knowledge structures needed in solving mathematical tasks and in terms of problem solving strategies. Changes in the individual's knowledge structures manifest themselves as quantitative and qualitative changes in the solution processes and in the strategies directing, guiding and controlling them. The individual's solution processes and strategies are accordingly those elements from which systematic knowledge is looked for and can be attained. The aim of the processual orientation could be defined as an attempt to create such a description system for each mathematical sub-area that ties together as truthfully as possible the calculative and conceptual aspects of mathematics learning. The task would be the description and explanation of the solutions of mathematical tasks or task types. As means it would employ the local models and theories of thought processes required by specific mathematical contents. I shall try to characterize the methods of the processual orientation as follows:

(a) Theoretical-conceptual analysis of the conceptual field studied (as a rule some narrow mathematical-logical content area) entailing essentially rational task analysis;
(b) Testing the developed hypothetical model or models in strictly controlled and replicable conditions;
(c) Perception, correction or even rejection of the hypothetical models based on empirical task analysis.

It is essential that we try to develop models which can be empirically tested. If the question is of numerical tasks presuming one operation, the models can be tested by the aid of the analysis based on quantitative reaction time (see for instance Ashcraft 1982; Groen & Parkman 1972; Parkman 1972). If we want to attain knowledge about the solution processes and strategies needed in uncommon mathematical problems, in verbal tasks or in numerical tasks involving many different mathematical-logical operations, the analyses based on reaction times are not valid any more. Subsequently, for instance, when trying to describe and to explain how pupils solve verbal arithmetic tasks, researchers have employed an interview where the pupil's talk, gestures, notings, and in some cases even his eye movements have been recorded (see for instance Carpenter & Moser 1982, 1984; Keranto 1983, 1985; Nesher et al 1982; Riley et al 1981). The studies in question have provided consistent, detailed and systematic information about how the most essential arithmetical tasks are solved, how solution strategies and processes develop and what kind of a knowledge structure is needed in the solutions of each task type. In fact, both the learning hierarchies of the behavioral orientation and the knowledge structures of Piaget's studies have been improved by the studies pursuing the orientation in question (see for instance Greeno 1978a,b; Klahr & Wallace 1976; Nesher et al 1982; Noelting 1980a,b; Resnick et al 1973, 1980; Riley et al 1981). Thus it can be argued that processually orientated research can provide a more truthful micro-level view about mathematics learning and thinking required by it than can the bahaviorally or structurally (in the Piagetian or gestalt psychological sense) orientated studies. What kind of a situation prevails in the studies dealing with the mathematics learning on the macro-level? In other words what do we know about the learning of more extensive mathematical learning entities and how has this been studied and should be studied? It is regrettable that research concerning the general means or learning strategies by which students try to analyze, interpret and recall broader subject areas, has orientated primaly on studying learning tasks in modern and humanistic subjects (see for instance Entwistle 1981; Marton & Säljö 1976a,b; Pask & Scott 1972; Pask 1976; von Wright et al 1979).

Furthermore, the terminology and methods used have been rather diverse as compared to the studies of micro-level mathematical tasks. I dare argue that the studies concerning the learning strategies of more extensive mathematical content areas have not yet proceeded beyond their primary stage. More work has been done and more progress has been made in the research of learning styles and on the psychometric orientation, which I shall deal with meet (look closer at learning styles for instance Leino, A-L. 1980; Leino, J. 1981; Messick 1976; Witkin et al 1977).

The psychometrically orientated research has had and still has a steady position in the research concerning mathematics learning and factors associated with it and affecting it. It is typical to the orientation in question to aim at identifying factors causing differences in mathematical achievements and to survey the correlative relations between these factors. The aim of psychometrically orientated researc' can be seen as the construction of such a description system th t reveals on the macro-level the factors that govern learning in each mathematical subject area and their interrelationships and influence channels. The task would be to describe and explain differences discovered in mathematics learning and in the solutions of mathematical tasks by operationalization and measurement of the factors hypothesized as causing these differences. The task is carried out by means of statistical techniques. I shall try to characterize the methods of the psychometric orientation in the following way:

(a) Modification of some more extensive research task of mathematics learning and teaching, formulation of the framework, problems and possible hypotheses based on theoretical scrutiny;
(b) Operationalization of the theoretical constructs represented in the theoretical framework, working out a research design and choosing the sample;
(c) Data collection using questionnaires, observation, interview, class examinations or test, written sources of information etc.;
(d) Quantification and statistical analysis of the data, as estimation of parameters, correlations, factor analysis, regression analysis, variance analysis etc.;
(e) Drawing conclusions and probably making corrections in the theoretical framework or even rejecting it.

If psychometrically orientated research is carried out in the way described above, it is in a way possible to avoid the critique focused, with apparently good reason, on the inductivistic branch of the psychometric orientation, where the empirical data are collected and analyzed without an adequate framework (cf. Niiniluoto 1983, 121-123). In addition psychometrically orientated studies have been critized of reificating social events and of giving too much emphasis to the methodological unity between natural sciences and social sciences (see for instance Markovic 1971). In other words, in the attempts to statistically describe learning and teaching, each examinee, a statistical unit, is attached to the disposition and characteristics supposed to be stable. Attempts are made to find the connections between these factors by statistical analysis. This is the case for instance with the structure of intelligence and its different faculties. In these studies the subjective intentions, beliefs, prevailing life situations and possible communicative difficulties of the persons studies are easily left without consideration (see a summary of ability structure studies, Leino 1981). This kind of objectifying and stamping is obviously characteristic of such studies of mathematics learning and teaching that use measurement, 'objective' statistical analyses, generalizations and predictions. On the other hand, psychometrically orientated research can be justified by its gradual reduction of the degree of idealization (see closer idealization, Niiniluoto 1980, 237-244; Niiniluoto 1983, 194; Niiniluoto 1986; Pietilä 1980). In other words, it is possible to proceed from the psychometrically orientated surveys to studies which are more detailed and which describe educational reality more accurately and in which the users and using situations of mathematical and natural languages are taken more and more into account. Thus we need such an orientation that makes it possible to take into account the whole interpretation system, i.e. what kind of messages are sent in the situations of mathematics learning and teaching and how these messages are interpreted by different participants (cf. hermeneutics and interpretation, Apel 1971; Lesche 1976; Niiniluoto 1983, 166-176; see also pragmatism, Niiniluoto 1980).

Accordingly such a research orientation that aims at taking into account learning content, the interactive process between student and teacher, at revealing faults and at changing educational practice in the direction of the goals, is called here the pragmatic orientation. The studies pursuing this orientation typically try to influence planning, the teaching and learning process in a way that the self-reflection of those participating in the research process increases and so that they might emancipate from distorted ideas, beliefs and attitudes. The aim of the pragmatic orientation would be to develop and promote mathematics learning and teaching in the direction of stated objectives. The task would be to advance the self-reflection of the participants and to free them from meaningless and trivial acts, beliefs and attitudes associated with mathematics learning and teaching. The task could be carried out by the aid of development or action research or as carefully designed teaching research (see action research, Cohen & Manion 1980, 174-189; Walker 1985; see also Galperin 1957; Galperin & Georgiev 1969). Subsequently the pragmatically orientated research emphasizes the aim of influencing student's intellectual development and learning in as natural a school environment as possible ('ecological validity'). As in the processual orientation the aim is to get to know the dynamics of student's mathematical thinking and learning employing mainly qualitative methods and by studying relatively small student groups. This is in accordance with the Piagetian tradition, which uses clinical interview as a primary means, as mentioned earlier. In fact clinical interview, combined with many other methods, has been used in the Soviet studies of mathematics learning and teaching (see Menchinskaya 1969). On the other hand, the Soviet studies that pursue the pragmatic orientation, emphasize the leading position and importance of teaching from the viewpoint of the pupil's intellectual development and learning. From the viewpoint of Piagetian research teaching is subjected to an dependent on the spontaneous developmental stages of the pupil. For instance Galperin's and Georgiev's studies refer to the fact that the 'misunderstandings' that have been noticed in the Piagetian studies concerning the initial learning phase of natural numbers are due to the cardinal emphasis appearing in the curriculum (Galperin & Georgiev 1969; see also the 'conservation' problems related to the learning of natural numbers; Piaget 1952; Flavell 1963; see also critique by Brainerd 1978, 1979; Keranto 1981, 1983, 1985; Klahr & Wallace 1976). In addition to this structurally and pragmatically orientated studies differ in the importance and position given to the use of natural and formal languages in mathematics learning and in the development of thinking. According to the pragmatic orientation the mathematics learning and thinking required by it, cannot be studied as separate from the meaning of natural and mathematical language and form the situations where they are used. One task of the pup[illegible]s to find the historically and socially determined objective meaning of the mathematical language. In fact mathematical language can be at first viewed as a vehicle of formulating mathematical constructs. It is only later that it becomes a real and valid indicator of those concepts (cf. Vygotsky 1962; see also Keranto 1981, 74-79). For instance Piaget's studies on number conceptualization are a very obvious opposite to the principle or commitment presented above, because the development of number conceptualization has been studied relatively isolated from the use of numerals and from the consideration of their different meanings. The main emphasis has been attached to studying the development of the learning of the structure of mathematical tasks and of the discovery of nonnumerical operations (see Brainerd 1978, 1979; Fuson & Hall 1982).

In the above I have tried to identify the research orientations of mathematics learning and teaching, to represent their goals and tasks and to compare their research findings and methods with each other. As a summary of the above I shall present the following table.

TABLE 1. Central research orientations, goals, tasks and means of the studies on mathematics learning and teaching

Behaviorally orientated research (behavioristic tradition)	Development of rehearsing systems for calculative skills in each content of mathematics teaching	Direction, control and prediction of the learning of calculative skills	Learning hierarchies: scaling and training studies
Structurally orientated research (Piagetian and gestalt psychological tradition)	Construction of the description system for the spontaneous development of mathematical-logical thinking	Description and explanation of the development of mathematical-logical thinking	Structu[illegible] models borrowed from mathematics or developed for the purpose in question: clinical interview
Processually orientated research (research of information processing, artificial intelligence)	Constructing such a description system for task solutions specif[illegible] [illegible]matical [illegible]nt that ma[illegible] possible to associate competently the calculative and conceptual aspects of mathematics learning	Description and explanation of a solution process and knowledge structure needed in the tasks of specific mathematical content areas	Local models of knowledge structures and performance processes in solving mathematical tasks: individual and/or group tests in strictly controlled conditions
Psychometrically orientated research (the tradition of differential psychology, survey-studies)	Identification of factors affecting achievements in mathematics learning and construction of macro-level description system of the relations between these factors	Macro-level description and explanation of the factors affecting the formation of performance differences	The framework represents the factors hypothesized to affect performance differences: survey and statistical analyses
Pragmatically orientated research (Soviet research tradition of mathematics learning and teaching, action research)	Construction of a description system that deals with the dynamics of learning and teaching of more comprehensive mathematical learning tasks	Changing educational practices in the direction of the goals so that the self-understanding of those participating in the research process increases and they free themselves from false deliefs, attitudes and meaningless acts	Teaching programs, curricula: studies of teaching, which may involve experimental designs & clinical interviews & observations, and action studies often associated with participating observation

JUSTIFICATION OF THE IDENTIFIED RESEARCH ORIENTATIONS

The preceding scrutiny may contribute to making the behaviorally, structurally and psychometrically orientated studies more explicit, complete and thorough by the processually orientated studies. Also pragmatically orientated research needs rational and empirical task analysis typical of the processual orientation dominant in the evaluation of teaching programs. Thus we end up in a dialogue about the justification of each research orientation or about why we need studies oriented in a certain way in order to promote didactic research and educational practice.

I think that the value of the behaviorally orientated research in mathematics learning and teaching is based on the fact that these studies have aimed at directing didactical research towards experimental and systematic research by using hypothetical learning hierarchies as a starting point and a guide in empirical data collection. From the viewpoint of practical values the studies orientated in the way in question may be tried to provide justification based on the merits they have in controlling and predicting the learning of mathematical skills. Thus the justification of the behaviorally orientated research is linked with the technical interest of knowledge.

Structurally orientated research may be justified by the merits it has in advancing the growth of systematic knowledge about the development of mathematical-logical thinking and in directing further studies into the conceptual and qualitative aspects of mathematics learning and teaching. The justification of structurally orientated research seems to be associated with the theoretical interest of knowledge. If the aim is to obtain truthful information about mathematics learning and teaching, both of the mentioned orientations can be regarded as one-sided. The behaviorally orientated research has one-sidedly concentrated on analysing and training calculative skills in mathematics learning. Structurally orientated studies have on the other hand, rather one-sidedly concentrated on defining and studying the development of nonnumerical logic operations presupposing the mathematical task solutions that require understanding.

Processually orientated research has aimed at eliminating the aforesaid biases by examining what is really taking place and what kind of knowledge structures are required in the task solutions of a specific mathematical content item. Besides this, processually orientated research is of a considerable importance in the systematization of scientific research, because it makes it possible to develop explicit performance and knowledge structure models so that these local models guide empirical data collection, analysis and interpretation of findings. This way the research projects which otherwise are easily inclined to remain isolated get a "common denominator" and they advance the systematic knowledge of the learning and teaching in each content area. In addition processually orientated research may be justifed by practical values. The aforesaid includes the notion of the "understading" and "liberating" aspects of the processual orientation, for instance when it aims at anticipating and remedying learning difficulties in mathematics and at planning mathematics teaching in the way that pays attention to significant and meaningful mathematics learning. The justification of processually orientated research is thus bound to pursuing as well theoretical as technical, hermeneutic and emancipatory knowledge interests.

Psychometrically orientated research may be justified by the fact that it has made it possible to identify the macro-level factors affecting mathematics learning. It has also made it possible to develop the system which to some extent describes the correlations between these factors. Justification can also be searched in the fact that the studies pertaining to the orientation in question provide insight into processually and pragmatically orientated studies to came.

The justification of pragmatically orientated research rests on the possibilities it offers for the description of situational dynamics in the teaching and learning of more extensive mathematical tasks. This implies practical aims and values, too. If the orientation in question makes it possible to free students and teachers from insignificant and meaningless activities and beliefs and to promote the growth of self-understanding of those participating in the research process, as many Soviet studies on mathematics learning and teaching imply, the orientation can be justified on the basis of hermeneutic and emancipatory knowledge interests.

In the above I have tried to identify the central research orientations of mathematics learning and teaching and the aims, tasks and methods associated with them. The presentation also includes discussion about the justification of each orientation. The reviews in question were preliminary surveys and they should be viewed as such. Many things still remain unclear. Were all the central orientations even identified? Was the view on the orientations or some of them too one-sided or reduced? How pure do the orientations in question appear in the present study of mathematics learning and teaching? Is it even reasonable to "dedicate oneself" to a specific research orientation? Would it not be profitable to use several different approaches when studying the problems of mathematics learning and teaching? Could the identified orientations not be regarded as complementary components of one comprehensive research program of mathematics learning and teaching? These are questions that should be answered in the future.

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SCHÜLERSCHWIERIGKEITEN IN DER ALGEBRA

Gustaf Adolf Lörcher

ZUSAMMENFASSUNG

Dieser Artikel ist ein Forschungsbericht. Das Ziel der Forschung ist die gewöhnlichsten algebraischen Schwierigkeiten bei den Schülern von Klassenstufen 7 und 8 in der Realschule zu finden.

Im ersten Teil werden Anforderungen im Algebraunterricht der Sekundarstufe I und Schülerschwierigkeiten in einem eingegrezten algebraischen Teilgebiet analysiert. Dann erfolgt die Schilderung über Konstruktion diagnostischer Tests.

Im dritten Teil werden die Testergebnisse von einige hundert Schülern gegeben (Schwierigkeitsfaktoren, durchschittlicher Zeitbedarf, Fehleranalyse).

Nach der Diskussion der Ergebnisse wird darauf Wert gelegt, wie die Ergebnisse Lehrern und Schülern beim Abbauen der algebraischen Schwierigkeiten helfen können.

EINLEITUNG

Am Umgang mit Variablen scheitern viele Schüler im Mathematikunterricht. In vielen Fällen gelingt es ihnen zwar, Überlebensstrategien zu entwickeln, mit denen sie bei Standardaufgaben einige massen zurechtkommen. Das Scheitern wird aber offensichtlich, wenn Anforderungen auftreten, in denen z.B. neben der Kenntnis von Umformungsregeln auch ein gewisses Variablenverständnis benötigt wird.

So wurde z.B. im CSMS-Projekt (Hart 1981) hur ein geringer Prozentsatz von Schülern ermittelt, der uber ein höheres Variablenverständnis verfügte. Ähnliche Phänomene ergab auch eine mehrjährige Auswertung der Schülerergebnisse bei der schriftlichen Realschulabschlussprufung in Mathematik in Baden-Württemberg. Auch nach 4-jähriger intensiver Beschäftigung mit Variablen scheiterten regelmässig rund 2/3 der Schüler and den Aufgabenteilen, in denen eine Beziehung mit Variablen formuliert und dann umgeformt werden sollte, ohne dass die Variablen durch konkrete Zahlenwerte ersetzt werden konnten.

Wo liegen die Ursachen für diesen Misserfolg? Welche Möglichkeiten bestehen, diesen Misserfolg abzubauen? Die Ursachen für den Misserfolg können im Stoff, beim Lehrer oder beim Schüler gesucht werden. Man kann einmal fragen, was spezifische stoffliche Schwierigkeitsfaktoren sind, die eine erfolgreiche Aneignung der Algebra erschweren. Sodann kann man versuchen, erfolgreiche und weniger erfolgreiche Vermittlungs- und Erklärungsansätze des Lehrers zu ermitteln. Schliesslich kann man danach fragen, welche allgemeineren Denkstrategien den häufigsten bei Schülern beobachteten Fahlern zugrunde liegen. Will man versuchen, im normalen Klassenunterricht durch den Lehrer realisierbare Möglichkeiten zum Abbau der Schwierigkeiten zu finden, so scheint es mir am erfolgversprechendsten (af Ebenstam & Nilsson 1979), zunächst nach den spezifischen stofflichen Schwierigkeiten der Algebra zu fragen, um dann zu untersuchen, welche Hilfsmittel dem Lehrer zur Verfügung stehen, um bei einzelnen Schülern zu diagnostizieren, welche stofflichen Schwierigkeiten zu Lernhindernissen werden und durch welche Massnahmen diese Hindernisse abzubauen sind.

Dazu werden im folgenden zunächst die stofflichen Anforderungen untersucht, denen der Schüler im Algebraunterricht begegnet. Dann wird in einem eingegrenzten algebraischen Teilgebiet (Addition/Subtraktion von Termen) analysiert, welche Schwierigkeiten vor allem auftreten, wie gross das Gewicht der einzelnen Schwierigkeitsfaktoren ist und was getan werden kann, um diese Schwierigkeiten abzubauen.

ANFORDERUNGEN IM ALGEBRAUNTERRICHT DER SEKUNDARSTUFE I

Nach dem in Baden-Württemberg gultigen Mathematiklehrplan fur Realschulen beginnt die systematische Beschäftigung mit der Algebra im 7. Schuljahr. Die folgende Übersicht (Tabelle 1) zeigt, um welche Anforderungen es sich hauptsächlich handelt, in welchen Stoffgebieten diese Anforderungen auftreten und welche Tätigkeiten dazu erforderlich sind.

TABELLE 1. Algebra im Mathematiklehrplan des 7. Schuljahres

Tätigkeiten / Stoffgebiete	umformen	einsetzen	zuordnen, herleiten	ablesen, einzeichnen
Terme	addieren/subtr. multiplizieren ausmultiplizieren und ausklammern (bin. Formeln) Bruchterme Wurzeln Potenzen	positive, negative Zahlen, Bruchzahlen in Variable eisetzen	verbale Ausdrucke in Terme übersetzen (z.B. Summe, gerade Zahl, zweistellige Zahl)	
Gleichungen	Termumformungen, Äquivalenzumformungen bei linearen Gleichungen, Bruchgleichungen Lösungsverfahren bei linearen Gleichungssyst. Lösung von quadratischen Gleichungen	Proble Einsetzugsverfahren Einsetzen in Lösungsformel	Aufstellen von Gleichungen in Textaufgaben (Zahlenrätsel, Bewegungsaufgaben, Verteilungs-, Mischungsrechnung)	grafische Lösung bei linearen Gleichungssyst. grafische Lösung von quadratischen Gleichungen
Formeln	Formel umstellen in -Geometrie -Algebra (bin. Formel) -Sachrechnen (z.B. Zinsformel)	Einsetzen von gegebenen und gesuchten Grössen oder von Termen in Formeln	Auswahl der richtigen Formel Zuordnung von gegebenen und gesuchten Grössen zu passenden Grössen in der Formel	Ablesen und Einzeichnen von Grössen in Skizzen
Funktionen	Umkehrfunktion, Nullstellen, Schnittpunkte bei linearen Funktionen Scheitelform bei quadratischen Funktionen	Wertetabelle, Punktprobe		Koordinatensystem, Ablesen, Einzeichnen im Koordinatensystem

Im Hinblick auf die Tätigkeiten geht es in den rechten beiden Spalten eher um Anwendungen algebraischer Fähigkeiten im Sachrechnen und in Geometrie; die eigentlich inneralgebraischen Tätigkeiten, das Umformen und Einsetzen sind in den linken Spalten nach Stoffgebieten aufgeschlüsselt. Betrachtet man die Stoffgebiete, so geht es bei Formeln und Funktionen wieder um vor allem um Anwendungen der Algebra in Sachrechnen und Geometrie, während die eigentlich inneralgebraischen Stoffgebiete bei Termen und Gleichungen zu finden sind.

Um spezifische algebraische Schwierigkeiten diagnostizieren zu können, erscheint es angebracht, in der Tabelle links oben zu beginnen und zunächst zu untersuchen, wo vor allem die Schwierigkeiten fur Schuler beim Umformen von Termen und Gleichungen liegen. Im folgenden wird über Untersuchungen berichtet, die sich auf Addition und Subtraktion von Termen beziehen.

Die Ursachen für die Schwierigkeiten können auf zwei verschiedenen Ebenen vermutet und gesucht werden: auf der Ebene des Variablenverständnisses wie auch auf der Ebene algorithmischer Regelkenntnisse im Umgang mit Variablen und Zahlen. Bei der im folgenden zugrundliegenden Analyse von schriftlichen Schülerleistungen lassen sich in erster Linie Aussagen über vorhandene oder fehlende bzw. gestörte Regelkenntnisse gewinnen. Es ist jedoch zu vermuten, dass das System der Regelkenntnisse beim Schüler umso leichter gestört wird oder durcheinander kommt, je weniger er sich auf sein Variablenverständnis verlassen kann. Das zeigt sich schon an der grundlegenden, immer wieder von Schülern geäusserten Frage "wenn ich die Grösse erst suche und sie nicht kenne, wie kann ich dann mit ihr rechnen?" Er lernt zwar, mit Buchstaben zu rechnen und auch ein Kette von Umformungsschritten mit Buchstaben als bedeutungsleeren Symbolen durchzuführen, ist sich seiner Sache aber nie ganz sicher, da häufig diese Frage unbeantwortet im Hintergrund steht, sobald er anfängt zu versuchen, seiner Rechnung mit Buchstaben einen Sinn zu geben.

Wie schwach sein System der Regelkenntnisse verankert ist, zeigt sich immer dann, wenn ungewohnte Situationen auftreten, in denen er das gewohnte Schema nicht mehr erkennen kann und deshalb ratlos vor der Frage steht "Kann das sein?" und "Was soll ich jetzt machen?"

Einige Beispiele sollen das verdeutlichen:

1) Im Bruchrechnen war der Schüler dazu angehalten worden, unechte Brüche jeweils in gemischte Zahlen zu verwandeln. Er hält sich auch in der Algebra daran und erhält Ausdrücke wie $5\frac{2}{3}x$. Wie soll er diesen Ausdruck deuten? Gehört das x zu 5 oder zu 2/3 oder zu beidem? Welche Verknüpfungszeichen muss er sich zwischen den 3 Termen denken?

2) Der Schüler hat gelernt, zur Sicherheit immer Klammern zu setzen. Er bekommt dadurch Ausdrücke wie $3\cdot(4\cdot 5)$. Er erinnert sich, dass man den Faktor 3 mit jedem Glied in der Klammer multiplizieren muss und rechnet $12\cdot 15$.

3) Der Schüler hat gelernt, dass das Ausrechnen der Klammer Vorrang vor allen anderen Rechnungen hat. In der Klammer steht $3x+5$. Also muss er hier etwas rechnen; z.B. $3x+5=8x$.

4) Bei der Lösung einer linearen Gleichung fällt plötzlich eine Seite weg. Wie soll jetzt weitergerechnet werden?

5) Bei der Lösung einer Gleichung hat der Schüler die Form $1{,}0x=-2$ erreicht. Wie soll er von da aus auf x kommen?

6) Der Schüler erkennt bei $3a+6a^2$ den gemeinsamen Faktor 3a und versucht auszuklammern: $3a(0+2a)$ oder $3a(+2a)$.

In allen diesen Fällen ist bemerkenswert, dass der Schüler vieles richtig weiss und sich an vieles richtig erinnert, dass ihm aber übergeordnete Kriterien fehlen, die es ihm ermöglichen würden, im Zweifelsoder Konflikt-fall, wenn er keine oder zu viele Regeln kennt, zu entscheiden, wie weiter zu verfahren ist. Da Variable für ihn häufig inhaltsleere Symbole oder blosse Fremdkörper sind, fehlt ihm die Fähigkeit, im Zweifelsfall z.B. einfache Zahlen an die Stelle der Variablen zu setzen, um so entscheiden zu können, welche der in Frage kommenden Möglichkeiten zu einem richtigen Ergebnis Führt.

Schülerschwierigkeiten bei algebraischen Umformungen können also sowohl auf der Ebene der Informationsaufnahme (falsche Wahrnehmung der für die Auswahl der richtigen Regel relevanten Merkmale einer Aufgabe; Störung der Wahrnehmung durch ungewohnte Form der Darstellung) als auch auf der Ebene der Informationsverarbeitung (fehlende oder falsche Regelkenntnis; fehlende oder falsche Kriterien für die Auswahl von Regeln aufgrund von Mängeln im Variablenverständnis) liegen.

Im folgenden soll über den Versuch berichtet werden, aus schriftlichen Schülerlösungen etwas über die wichtigsten Schülerschwierigkeiten, die bei algebraischen Umformungen auftreten, zu entnehmen. Dieser Ansatz hat den Nachteil, dass man nur das schriftliche Produkt, nicht aber den zugrundeliegenden gedanklichen Prozess analysieren kann und deshalb z.T. nur Vermutungen über die zugrundliegenden Ursachen aufstellen kann. Der Vorteil dieses Ansatzes ist, dass man Informationen nicht nur über einzelne Schüler, sondern über ganze Klassen erhält und dass dabei solche Daten analysiert werden, wie sie laufend im Unterricht anfallen. Die Chance ist deshalb gross, dass die gewonnenen Erkenntnisse sich vom Lehrer leichter auf die Diagnose und Therapie der in der eigenen Klasse auftretenden Schwierigkeiten übertragen lassen.

SCHÜLERSCHWIERIGKEITEN BEI DER ADDITION UND SUBTRAKTION VON TERMEN

Schwierigkeitsanalyse

Welche Schwierigkeiten können z.B. bei der Umformung eines Terms wie a·(-3) + 4b - (a/2-b)·2 auftreten?

Beobachtet man einzelne Schüler, so wird man bei jedem Schüler verschiedene individuelle Probleme feststellen. Darüber hinaus gibt es aber eine Reihe von generellen Schwierigkeitsmerkmalen, die weniger mit dem einzelnen Schüler, als mit den Besonderheiten dieser Aufgabe zu tun haben.

Diese Schwierigkeiten können einmal mit dem besonderen Charakter algebraischer (im Unterschied zu arithmetischen) Operationen zu tun haben, sie können zum anderen mit den Konventionen zusammenhängen, deren Kenntnis in der Algebra stillschweigend vorausgesetzt wird, und sie lassen sich schliesslich in einzelne Faktoren zusammenfassen und ordnen. Diese Isolierung einzelner Schwierigkeitsfaktoren bietet anschliessend die Grundlage für die Konstruktion von diagnostischen Tests.

Schwierigkeiten bei algebraischen Operationen

In der Algebra muss der Schüler an entscheidenden Stellen umlernen. Er muss gewohnte und richtige Bedeutungen und Handlungsanweisungen, die er sich in der Arithmetik angeeignet hat, in der Algebra nicht nur modifizieren, sondern z.T. vollständig aufgeben und durch neue ersetzen. Einige Beispiele:

1) Die Operationszeichen "+" und "-" verlieren in der Algebra den aus der Arithmetik gewohnten operativen Sinn. Während Sie früher die Aufforderung für die Ausführung einer Rechnung darstellten, muss der Schüler jetzt erkennen, dass er z.B. bei -3a+4b nichts rechnen darf.

Ihm wird also ein Eckpfeiler seines bisherigen Orientierungssystems entzogen.

2) Während der Schüler bisher aus der Angabe der Operationszeichen erkennen konnte, welches Endprodukt gefordert war, lässt sich jetzt nicht mehr eindeutig erkennen, ob z.B. bei obiger Aufgabe das Ergebnis -4a+6b genügt, oder ob am Schluss noch ausgeklammert werden muss: 2(-2a+3b) oder -2(2a-3b).

3) Klammern bedeuteten bisher für den Schüler, dass die Operation innerhalb der Klammer zuerst auszuführen war. Jetzt muss er lernen, dass häufig innerhald der Klammer nichts gerechnet werden darf, z.B. bei a/2-b; ausserdem muss er lernen, dass das mühsam erlernte Distiributivgesetz (jedes Glied innerhalb der Klammer mit dem Faktor ausserhalb multiplizieren) nicht zutrifft, wenn innerhalb der Klammer nur Produkte stehen.

4) Er muss häufiger als bisher vor Beginn der Rechnung untersuchen und unterscheiden, ob es sich um einen Summenterm oder Produktterm handelt; z.B. a·(-3) im Unterschied zu a-3.

5) Er muss erkennen, was vertauscht werden kann und was nicht; z.B. -3a+4b = 4b-3a, nicht aber = 3a-4b.

6) Im Unterschied zum bisher bei Masseinheiten gewohnten Umgang mit Buchstaben bedeuten jetzt verschiedene Buchstaben nicht notwendig verschiedene Grössen. Während die Angabe s=h bisher falsch und die Angabe m=g bisher unsinnig war, kann beides innerhalb der Algebra richtig sein.

Schwierigkeiten durch Konventionen

Der Umgang mit Variablen wird dadurch weiter erschwert, dass sich der Schüler viele Konventionen merken muss, die oft im Unterricht kaum kenntlich gemacht und wie selbstverständlich vorausgesetzt werden.

- So muss sich der Schüler zwischen den vielen Möglichkeiten zurechtfinden, wann man welche Zeichen weglassen kann und wann nicht. So wird "1" als Faktor oder Divisor meistens weggelassen, muss aber geschrieben werden, wenn 1 als alleinige Zahl beim Ausklammern übrig bleibt oder alleiniger zähler ist.
"+" wird als Vorzeichen oder als Operationszeichen zwischen ganzer Zahl und Bruch weggelassen, muss aber als Operationszeichen zwischen zwei sonstigen Termen geschrieben werden.
"mal" wird zwischen Variablen, zwischen Zahl und Variable und zwischen Zahl oder Variablen und Klammer i.a. weggelassen, bei umgekehrter Reihenfolge (Zahl hinter der Variablen, Zahl oder Variable hinter der Klammer) oder zwischen Klammern geschrieben.Klammern um gemischte Zahlen, um Zähler und Nenner von Brüchen werden weggelassen, mussen aber gesetzt werden, wenn erweitert oder gekürzt oder wenn in ein Produkt eingesetzt wird.

- Obwohl die Reihenfolge bei Termen beliebig ist, muss der Schüler lernen, wie zu ordnen ist; im Produkt: zuerst Vorzeichen, dann Koeffizient, dann Variable in alphabetischer Reihenfolge (wobei es in Bruchschreibweise sowohl a/2 wie auch 1/2 a heissen kann); bei Summen; nach der Summe der Hochzahlen der Variablen und bei gleicher Summe: alphabetisch.

Schwierigkeitsfaktoren

Versucht man die einzelnen Schwierigkeiten zu isolieren, so kann man vermuten, dass Addition und Subtraktion von Termen durch das Auftreten jedes der folgenden Schwierigkeitsmerkmale erschwert wird:

- Zahlbereich (je grösser die Zahlen, Auftreten von Komma, negatives Vorzeichen, Bruchstrich)
- Umfang (Anzahl der Summanden, Anzahl der Faktoren in den zu addierenden Produktermen)
- Prägnanz (ungeordnete Reihenfolge der Summanden, ungeordnete und z.T. verschiedene Reihenfolge innerhalb der Produktterme, Verschiedenartigkeit der Summanden)
- Klammern (Assoziativgesetz der Addition, Distributivgesetz,

Assoziativgesetz der Multiplikation; Klammern um negative Zahlen)

- Sonderfälle (Terme z.T. ohne Koeffizienten; Null als Summand, als Faktor, Null als Ergebnis).

Konstruktion diagnostischer Tests

Die Konstruktion diagnostischer Tests erfolgte mit Hilfe einer Diagnosematrix. Nach Auswahl der relevantesten Schwierigkeitsfaktoren wurden in dieser Matrix zu den wichtigsten Kombinationen der Schwierrigkeitsfaktoren Aufgaben konstruiert. Vorbild dafür war das im PUMP-Projekt von W. Kilborn und B. Johansson in Göteborg für Grundrechenarten entwickelte Verfahren, das in der Zwischenzeit gemeinsam mit D. Gerster weiterentwickelt und z.B. auch für die Konstruktion von diagnostischen Tests in der Bruchrechnung erfolgreich eingesetzt worden war (Lörcher 1982).

Nach einer Reihe von Voruntersuchungen und sich daran anschliessenden Revisionen kam folgende Diagnosematrix zustande.

TABELLE 2. Diagnosematrix

Schwierigkeit faktoren	N mit Koeffiz.	N z.T. ohne Koeffiz.	Z mit Koeffiz.	Z z.T. ohne Koeffiz.
Normalfal	5a+2a *6ab-4ab*	4a-a *ab+8ab*	*-5a-2a* -6ab-4ab	*-4a+a* -ab-8ab
Null im Ergebnis	*1b-b 1* bc 1-1bc	*b-1b* 1bc-bc	-1b+b 1 *-bc 1+1bc*	[illegible]+1b *-1bc+[illegible]c*
Null als Faktor	0 c+9c *7cd-0*	c-0 *0 cd+cd*	*-0c-9c* -7cd+0	*-c+0* -0cd-cd
versch.artige Terme	*3e-8+2e* 4f+2e-3f	*5-e-1* 6e+5f+e	-3e+8+2e *-4f-2e+3f*	-5+e+1 *-6e-5f-e*
verschiedene Reihenfolge	5x+x 3 *xy 6-2yx*	x 5-x *yx+xy*	*-5x-x 3* -xy 6+2yx	*-x 5+x* -yx-xy
Klammern	*8y+(2y+4y)* (6yz-4yz)+3yz	*(5y-3y)-y* 9yz-(yz+3yz)	-8y-(2y+4y) *(6yz-4yz)-3yz*	-(5y-3y)+y *-9yz+(yz+3yz)*

Erläuterungen:

- Die durch Wahl des Zahlbereichs verursachten Schwierigkeiten wurden auf den vergleich des Rechnens mit natürlichen (N: Spalte 1 und 2) und des Rechnens mit negativen Zahlen (Z: Spalte 3 und 4) beschränkt. Dabei schien es - wie aus Voruntersuchungen hervorging - weder in N noch in Z eine Rolle zu spielen, ob es sich um Addition oder Subtraktionen handelte. Die Art der Operation wurde deshalb nicht berücksichtigt. Ebenfalls nicht berücksichtigt wurden grosse Zahlen, Kommazahlen und Brüche, da zu erwarten war, dass dadurch lediglich etwas über arithmetische Schwierigkeiten, aber nichts wesentlich Neues über algebraische Schwierigkeiten zutage treten würde. Die Koeffizienten wurden demgemäss so gewählt, dass nur mit einstelligen Zahlen gerechnet werden musste.

- Beim Umfang wurde nur die Anzahl der Faktoren in den Produkten berücksichtigt (1. Aufgabe in jedem Feld mit einer, 2. Aufgabe mit zwei Variablen). Die Anzahl der Summanden wurde bei den Aufgaben mit gleichartigen Termen ohne Klammer auf 2 beschränkt, da sich keine wesentlichen Unterschiede im Vergleich zu Termen mit 3 Summanden ergeben hatten.

- Bei der Prägnanz wurde die z.T. verschiedene Reihenfolge innerhalb des Produktterms (Zeile 5 im Vergleich zu Zeile 1) und die Verschiedenartigkeit von summanden (Zeile 4 im Vergleich zu Zeile 1) berücksichtigt.

- Bei Klammern (letzte Zeile) wurde das Distributivgesetz nur insofern berücksichtigt, als auch Terme mit einem "-" vor der Klammer vorkamen. Da es sich immer um gleichartige Terme handelte, konnten die Schüler sowohl zunächst innerhalb der Klammer rechnen als auch die Klammer auflösen oder weglassen (Assoziativgesetz der Addition).

- Von den Sonderfällen wurde einmal die Schwierigkeit z.T. fehlender Koeffizienten (Spalte 2 und 4 im Vergleich zu Spalte 1 und 3) berücksichtigt wie auch die Sonderfälle mit der Null als Ergebnis (Zeile 2 im Vergleich zu Zeile 1) sowie als Faktor oder Summand (Zeile 3 im Vergleich zu Zeile 1).

Um den Test nicht zu umfangreich werden zu lassen, wurden die 48 Aufgaben so auf 2 Testformen verteilt, dass möglichst gleichwertige Tests entstanden (Testform A oben in Normalschrift, Testform B kursiv).

Die Beschränkung des Umfangs des Tests brachte es mit sich, dass nur 3 Schwierigkeitskategorien voll miteinder kombiniert werden konnten (N/Z mit/z.T. ohne Koeff. in der Kopfzeile und Anzahl der Variablen in jedem Feld); die übrigen Schwierigkeitskategorien (siehe Randspalte) können nur mit dem Normalfall (1. Zeile) verglichen werden.

Empirische Ergebnisse

Gesamtergebnisse

Der vorliegende diagnostische Test wurde im Juli 1984 in 2 siebten und 5 achten Realschulklassen in Baden-Württemberg von insgesamt 171 Schülern bearbeitet.

Der durchschnittliche Fehlerprozentsatz betrug

- im A-Test	19,5 % ± 13,6 %
- im B-Test	14,9 % ± 12,5 %
- insgesamt	17,2 % ± 12,1 %

Die Korrelation zwischen A- und B-Test lag bei .71.

Ähnliche Ergebnisse wurden mit der Vorform dieses Tests im Juni 1982 in 7 achten Realschulklassen mit insgesamt 168 Schülern erzielt (A-Test: 19,4 ± 13,9 %; B-Test: 15,9 % ± 13,5 %; insgesamt: 17,6 % ± 13,0 %; Korrelation: .77).

Der durchschnittliche Zeitbedarf pro Aufgabe (in Klammern die entsprechenden Ergebnisse von 1982) betrug

- im A-Test	16,7s ± 5,9s	(15,6s ± 6,4s)
- im B-Test	11,6s ± 4,1s	(11,3s ± 4,0s)
- insgesamt	14,2s ± 4,4s	(13,5s ± 4,2s)

Die Korrelation zwischen zeitbedarf im A- und B-Test lag bei .52 (.28), die Korrelation zwischen Fehler und Zeitbedarf lag bei .27 (.30). Insgesamt verringerte sich also die Fehlerzahl vom A-Test zum unmittelbar anschliessend bearbeiteten B-Test um rd. 20 bis 25 % und der Zeitbedarf ging um rd. 25 bis 30 % zurück.

Schwierigkeitsfaktoren

Bei den einzelnen Aufgaben traten im Haupttest 1984 folgende Fehlerprozentsätze auf (A-Test: Normalschrift, B-Test: kursiv):

TABELLE 3. Fehlerprozentsätse

Schwierigkeits-faktoren	N mit Koeffiz.		N z.T. ohne Koeffiz.		Z mit Koeffiz.		Z z.t ohne Koeffiz.	
Normalfall	5a+2a *6ab-4ab*	1% *2%*	4a-a *ab+8ab*	4% *4%*	*-5a-2a* -6ab-4ab	*15%* 20%	*-4a+a* -ab-8ab	*16%* 26%
Null im Ergebnis	*1b-b 1* bc 1-bc	*9%* 12%	*b-1b* 1bc-bc	*15%* 14%	-1b+b 1 *-bc 1+bc*	13% *12%*	-b+1b *-1bc+bc*	12% *16%*
Null als Faktor	0 c+9c *7cd-0*	15% *1%*	c-0 *0 cd+cd*	15% *18%*	*-0c-9c* -7cd+9	*17%* 3%	*-c+9* -0cd-cd	5% 36%
versch.artige Terme	*3e-8+2e* 4f+2e-3f	5% 4%	*5-e-1* 6e+5f+e	*13%* 5%	-3c+8+2e *-4f-2e+3f*	15% *23%*	-5+e+1 *-6e-5f-e*	33% *19%*
verschiedene Reihenfolge	5x+x 3 *xy 6-2yx*	9% *16%*	x 5-x *xy+xy*	4% *20%*	*-5x-x 3* -xy 6+2yx	*20%* 33%	*-x 5+x* -yx-xy	*18%* 40%
Klammern	*8y+(2y+4y)* (6yz-4yz)+3yz	*8%* 9%	*(5y-3y)-y* 5yz-(yz+3yz)	*23%* 35%	-8y-(2y+4y) *-(6yz-4yz)-3yz*	44% *42%*	-(5y-3y)+y *-9yz+(yz+3yz)*	48% *21%*

Vergleicht man die beiden Aufgaben in jedem Feld sowie die entsprechenden Spalten und Zeilen, so kann man abschätzen, welches Gewicht die einzelnen Schwierigkeitsfaktoren haben, oder anders ausgedrückt, welche Reduktion des Lösungsprozentsatzes bei Auftreten eines bestimmten Schwierigkeitsfaktors in einer Aufgabe zu erwarten ist. Gleichzeitig kann man erkennen, ob die einzelnen Schwierigkeitsfaktoren im wesentlichen unabhängig voneinander sind, bzw. wo sie sich bei gemeinsamem Auftreten verstärken oder abschwächen. Schliesslich kann man die Aufgaben noch genauer unter die Lupe nehmen, bei denen der Fehlerprozentsatz unerwartet hoch oder unerwartet niedrig ist.

Die beiden gewichtigsten Schwierigkeitsfaktoren scheinen negative Zahlen (Z: Spalte 3 und 4) und Klammern (letzte Zeile) zu sein. Bei ihrem Auftreten verringert sich der Lösungsprozentsatz jeweils um rund 20 %. Bei den negativen Zahlen fällt auf, dass sie bei Auftreten der Null nur wenig zusätzlich erschwerend wirken. Vergleicht man die einzelnen Aufgaben mit Klammern, so fällt auf, dass sich bei einem "Minus" vor der Klammer die Schwierigkeiten von Klammern und negativen Zahlen verstärken. Die Schüler scheinen demnach nicht zuerst die Summen oder Differenzen in der Klammer zu berechnen, sondern versuchen offensichtlich, die Klammer sofort aufzulösen, und scheitern dann besonders häufig, wenn ein Minus vor der Klammer steht.

Verschiedene Reihenfolge der Faktoren innerhalb der Produkte scheint den Lösungsprozentsatz durchschnittlich um rund 10 % zu senken. Dabei ist es für den Schüler deutlich schwerer, wenn 2 Variable mit vertauschter Reihenfolge auftreten, als wenn nur mit einer Variablen und vertauschtem Koeffizienten zu rechnen ist.

Tritt eine Null im Ergebnis (2. Zeile) oder als Faktor (3. Zeile) auf, so scheint der Lösungsprozentsatz im Vergleich zu Termen mit natürlichen Zahlen (Zeile 1, 1. und 2. Feld) um rund 10 % zu sinken. Eine Null als Summand schafft dagegen kaum zusätzliche Schwierigkeiten.

Der Lösungsprozentsatz erniedrigt sich um rund 5 %, wenn Koeffizienten z.T. fehlen (2. und 4. Spalte im Vergleich zur 1. und 3.) sowie wenn in einer Summe Verschiedenartige Terme auftreten. Terme mit z.T.

fehlenden Koeffizienten scheinen zu zusätzlichen Schwierigkeiten zu führen, wenn Klammern auftreten; bei verschiedenartigen Termen treten verstärkt Schwierigkeiten auf, wenn es sich bei den Summanden um Variablenterme und blosse Zahlen handelt.

Vergleicht man die Terme, in denen zwei Variable vorkamen, mit den Termen mit einer Variablen, so war eine Erschwerung nur dann festzustellen, wenn die Variablen im Produkt ungeordnet in verschiedener Reihenfolge vorkamen.

Fehleranalyse

Die von den Schülern gemachten Fehler kann man in arithmetische und algebraische Fehler einteilen und dabei folgende Fahlertypen unterscheiden.

(1) Arithmetische Fehler

Z-Fehler: Fehler durch Mängel beim Rechnen in Z.
Dabei handelt es sich hauptsächlich

- um falsche Vorzeichen; Bsp.: -ab-8ab=9ab (15 % aller Schüler)
- oder um falsche Operationen (v.a. Verwechslung von Addition und Subtraktion in Z); Bsp.: -ab-8ab=7ab (5 %) oder = -7ab (3 % aller Schüler).

N-Fehler: Fehler beim Rechnen mit Null (bzw. mit dem neutralen Element). Am häufigsten kamen hier vor:

- Verwechslung von 0 mit 1: Ocd+cd=2cd (13 % aller Schüler)
- Verwechslung von 1 mit 0: -1bc+bc=bc (4 % aller Schüler)
- oder es wurde nicht vollständig zu Ende gerechnet: b-1b=0b (9 %).

Weitere mögliche arithmetische Fehler (Rechenfehler beim Rechnen mit natürlichen Zahlen, Kommafehler, Bruchrechenfehler) wurden durch Beschränkung der Koeffizienten auf einstellige ganze Zahlen weitgehend ausgeschlossen.

(2) Algebraische Fehler

V-Fehler: Variablenfehler (durch verschiedene Behandlung von Zahlen und Variablen). Am häufigsten war die Deutung der

- Subtraktion von Variablen als Wegnehmen: 1bc-bc=1 (6 % der Schüler)
- Addition von Variablen als Hinzufügen: -5+e+1=-4e (9 % der Schüler).

L-Fehler: Lücken durch fehlende Umformung (vor allem wenn nicht erkannt wurde, dass Terme zusammengefasst werden konnten). Die Umformung fehlte häufig

- bei verschiedenartigen Termen: yx+xy=yx+xy (11 % der Schüler)
- bei der Null: c-0=c-0 (10 % der Schüler).

K-Fehler: Klammerfehler.

Hier waren am häufigsten:

- bei "-" vor der Klammer falsches Umdrehen der Vorzeichen in der Klammer: -(5y-3y)+y=9y (11 % der Schüler)
- Nichtberücksichtigen der Klammer: -(5y-3y)+y=-7y (7% der Schüler).
- falsche Hierarchie der Operationen: 5x+x·3=6x·3 oder =18x (3 %)
- Verwechslung von Assoziativgesetz und Distributivgesetz: 8y+(2y+4y)=22y (1%).

O-Fehler: Operationsfehler durch Verwechslung von "plus" mit "mal". S: Sonstige Fehler waren vor allem Rechenfehler.

Traten innerhalb einer Termumformung mehrere Fehler auf, so wurde jeweils nur der Hauptfehler registriert. Das hiess z.B. bei Zusammentreffen von arithmetischem und algebraischem Fehler, dass nur der algebraische Fehler erfasst wurde.

Die folgende Grafik zeigt, welche Unterschiede in der Häufigkeitsverteilung der Fehler auftraten, wenn man jeweils die Aufgaben, die einen bestimmten Schwierigkeitsfaktor enthielten, mit den entsprechenden Aufgaben ohne diesen Schwierigkeitsfaktor vergleicht. Links sind die Schwierigkeitsfaktoren dargestellt, die im Test mit allen anderen Faktoren kombiniert wurden (Spalten und Felder der Diagnosematrix), rechts die Schwierigkeitsfaktoren, die nur mit den links dargestellten Faktoren kombiniert wurden (Zeilen der Diagnosematrix). Rechts unten kann man die Fehlerverteilung im A-Test mit der im unmittelbar anschliessend bearbeiteten B-Test vergleichen.

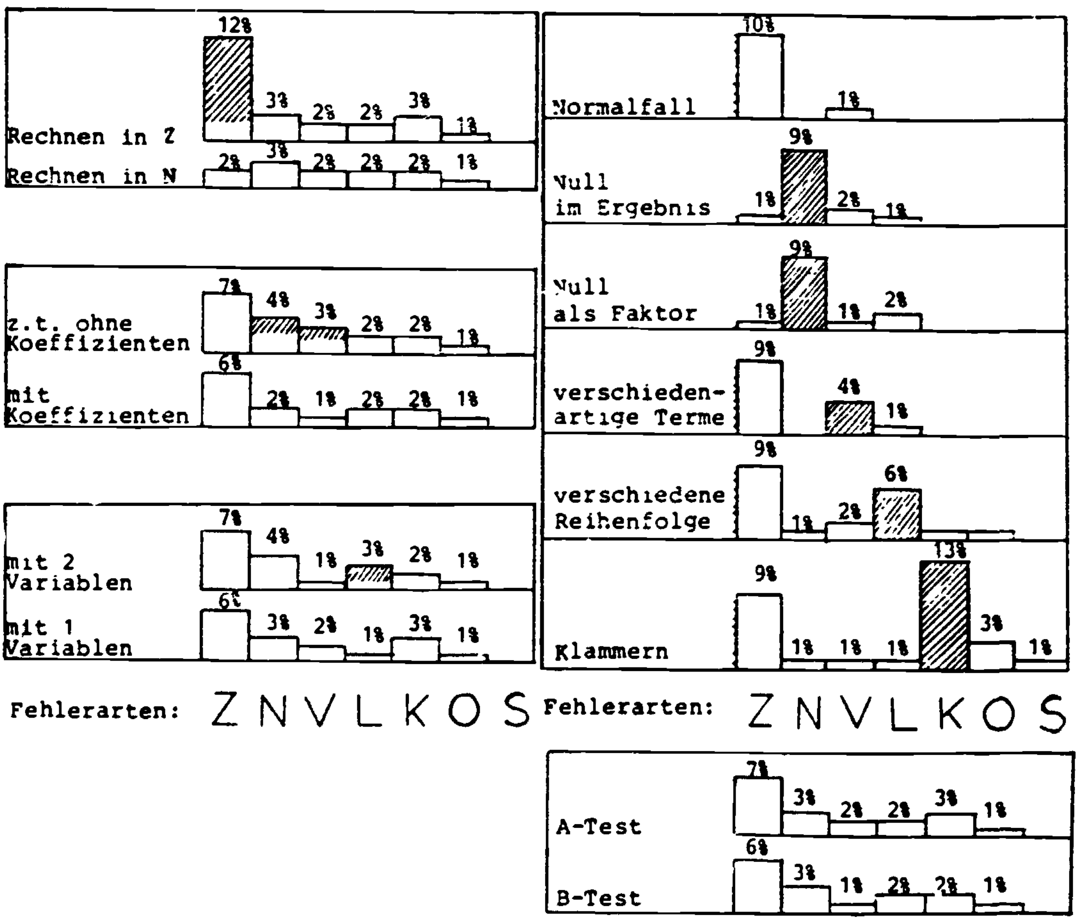

ABBILDUNG 1. Fehlerverteilung

Beim Rechnen in Z spielten erwartungsgemäss die Z-Fehler die grösste Rolle; sie machten hier mehr als die Hälfte aller Fehler aus. Es fällt auf, dass die Unsicherheit im Umgang mit negativen Zahlen auch die Rechensicherheit bei natürlichen Zahlen zu beeinträchtigen schien (immerhin durchschnittlich bei 2 % der Schüler).

Die durch z.T. fehlende Koeffizienten hervorgerufene Schwierigkeit schlug sich vor allem in einer Erhöhung der Nullenfehler und der Variablenfehler nieder.

Bei Termen mit 2 Variablen steigt der Anteil der Lückenfehler, da der Schüler häufiger die Gleichartigkeit von zwei Summanden nicht erkennt, bei denen die Variablen in verschiedener Reihenfolge als Faktoren vorkommen.

In der rechten Spalte wird sichtbar, dass bei Aufgaben mit Null (als Ergebnis oder als Faktor bzw. Summand) nahezu jeder 10. Schüler am Rechnen mit Null scheitert. Z-Fehler spielen bei Aufgaben mit Null erwartungsgemäss so gut wie keine Rolle.

Verschiedenartige Terme führen zu mehr Variablenfehler (falsche Verknüpfung von Zahl und Variable); verschiedene Reihenfolge der Faktoren hat eine Erhöhung der Lückenfehler zur Folge und bei Klammeraufgaben nehmen neben den Klammerfehlern auch die Operationsfehler zu (Multiplikation statt Addition); Klammern scheinen demnach für etliche Schüler ein fester Auslöser fürs Distributivgesetz zu sein.

Vergleicht man schliesslich die Fehlerverteilung bei A-Test und B-Test, so sieht man eine annähernd gleiche Verteilung; immerhin ist aber nach einer Aufwärmphase (A-Test) im Hinblich auf das Rechnen mit negativen Zahlen, mit Klammern und auf den Umgang mit Variablen ein gewisser Lerneffekt beim Schüler zu beobachten (Rückgang der Z-, K- und V-Fehler) auch ohne zusätzliche Instruktionen.

Diskussion der Ergebnisse

Gesamtergebnisse

Die erzielten Lösungsprozentsätze von über 80 % scheinen auf den ersten Blick einen relativ grossen Erfolg des Algebraunterrichts zu signalisieren. Beachtet man aber, dass hier nur eine Elementaroperation untersucht wurde und dass es sich meist um Achtklässler handelte, also Schüler, die sich schon im 2. Jahr intensiv mit algebraischen Umformungen beschäftigten, dann sieht man, dass der erreichte Stand nicht befriedigen kann. Sind z.B. bei der Lösung einer Gleichung mehrere Additionsumformungen durchzuführen, so muss man damit rechnen, dass allein dadurch bedingt, der Lösungsprozentsatz nicht über 50 % bis 60 % steigen wird ($0{,}8^2$ bzw. $0{,}8^3$).

Der durchschnittliche Zeitbedarf von 15s±5s gibt dem Lehrer wenigstens einen Anhaltspunkt dafür, wieviel Zeit er dem Schüler z.B. bei Klassenarbeiten mindestens pro Termumformung zugestehen sollte. Will er einigermassen sicher sein, dass so gut wie alle Schüler mit der Zeit zurechtkommen, so sollte er pro Termaddition oder -subtraktion mindestens 25s ($\bar{x}+2s_x$) veranschlagen. Die geringe Korrelation zwischen Fehlerzahl und Zeitbedarf zeigt zudem, dass Zeitknappheit in Klassenarbeiten die Konsequenz hat, dass ein grösserer Teil der Schüler (langsame, gründliche Schüler) ihr vorhandenes Wissen nicht zeigen können.

Die vom A-Test zum B-Test eingetretene Verbesserung unterstreicht die Wichtigkeit einer Anwärmphase für den Schüler, da sie es ihm offensichtlich eher ermöglicht, von früher vorhandenes Wissen wieder zu aktualisieren.

Schwierigkeitsfaktoren

. er fällt auf, wie gross die durch negative Zahlen in der Algebra verursachten Probleme sind. Das dichte Aufeinanderfolgen der beiden im 7. Schuljahr neu eingeführten Stoffgebiete verstärkt offensichtlich die Gefahr, dass der Schüler an der Algebra scheitert, weil er das vorhergehende Gebiet des Rechnens mit negativen Zahlen nicht gemeistert hat. Es ist zu vermuten, dass sich diese Probleme noch verstärken, wenn wie bei Gleichungen zur Addition und Subtraktion auch noch die Multiplikation und Division mit negativen Zahlen in der Algebra kommt.

Die grossen durch Klammern verursachten Schwierigkeiten machen deutlich, dass dem Umgang mit diesen Symbolen im Algebra mehr Aufmerksamkeit gewidmet werden muss. Insbesondere scheint zu wenig geklärt zu werden, wann man auch weiterhin (wie in der Arithmetik) zuerst innerhalb der Klammer rechnen kann. Die Gefahr, dass Klammern zum automatischen Auslöser fur Anwendung des Distributivgesetzes werden, macht sich später vor allem bei Klammern in Produkten bemerkbar, wenn der Schüler z.B. rechnet $(8 \cdot 8 \cdot \pi):4 = 2 \cdot 2\pi$

Die Null als besonderer Schwierigkeitsfaktor scheint den Schüler durch die ganze Schulzeit zu begleiten. Dass diese Schwierigkeiten nicht erst hier oder beim Bruchrechnen (Lörcher 1982), sondern massiv als Hauptfehler schon beim Einmaleins auftreten (Lörcher 1983, 1985), deutet darauf hin, dass von Beginn an im Mathematikunterricht das Rechnen mit Null dem Schüler oft falsch (Null ist nichts) oder nicht plausibel gemacht wird.

Fehleranalyse

Die bei der Untersuchung der Fehlerverteilung deutlich gewordene Konzentrierung einzelner Fehlerarten auf einzelne Aufgabentypen zeigt, dass es gelingen kann, durch sorgfältige Kontrolle der Schwierigkeitsmerkmale einer Aufgabe die für den Schüler wirksam werdenden Schwierigkeiten effektiv voneinander zu isolieren. Die

Konstruktion der Aufgaben mit Hilfe einer Diagnosematrix hat sich dabei als brauchbares, weil für den Lehrer handhabbares Hilfsmittel erwiesen.

Von den postulierten Schwierigkeitsfaktoren haben sich dabei das teilweise Fehlen von Koeffizienten und das Auftreten von zwei Variablen in einem Term als weniger ins Gewicht fallende und nicht so deutlich zu isolierende Schwierigkeiten erwiesen, während die anderen sich trennscharf in je einem verschiedenen Fahlertyp beim Schüler niederschlugen.

Konsequenzen

Für die Diagnose

Verzichtet man auf eine feinere Typisierung der Fehler, so kann man sich in einer Klasse relativ rasch ohne viel zusätzlichen Arbeitsaufwand durch Eintragen der bei der Korrektur des diagnostischen Tests pro Schüler und pro Aufgabe festgestellten Fehlertypen in eine Klassenliste einen raschen Überblick verschaffen, bei welchen Aufgaben und bei welchen Schülern sich welche Schwierigkeiten konzentrieren. Die folgende Tabelle zeigt einen Ausschnitt aus einer Klassenliste mit den Ergebnissen 8 verschiedener Schüler:

TABELLE 4. Addition und Subtraktion von Termen: Klassenliste

Schulart: Realschule
Ort:
Schule: n=22
Klasse: 8 Stunde: 3.
Datum: 25.10.1984

	1	2	3	4	5	6	7	8	K	L	N	O	R	V	Z	
Σt																
tA	10	10	10	10	10	10	10	7								
tB																
m/W	W	m	m	W	W	m	W	m								
Nr	1	2	3	4	5	6	7	8	K	L	N	O	R	V	Z	
A 1 a) 5a+2a = 7a	V				R		V						4	4	1	9
b) bc 1-1bc = 0	V	N	N			N		N			11			1		12
c) 0 c+9c = 9c	V				N		V				3			9		12
d) 4f+2e-3f = 2e+f	V	Z	V		V		V							18		19
e) 5x+x 3 = 8x	V				R		V						3	3		6
f) (6yz-4yz)+3yz=5yz	V				R								1	1		2
Fehlersumme 1	6	2	2	-	5	1	4	1	-	-	14	-	8	36	2	60
2 a) -8y-(2y+4y) = 14y	L		Z	Z	Z	Z	V		1	1			2	2	13	19
b) -xy 6+2yx = -4xy	Z			Z		Z	L			1				3	11	15
c) -3e+8-2e = 5e+8	V	V	V	Z	V	V	V	V						21	1	22
d) -7cd+0 = 7cd			Z				V							9	1	10
e) -1b+b 1 = 0	V	N	N		N	Z	V	N			7			5	5	17
f) -6ab+4( ) = -2ab	V	Z			V	Z	V							4	9	13
Fehlersumme 2	5	3	4	3	4	5	6	3	1	2	7	-	2	44	42	96
3 a) 9yz-(yz+3yz)=5yz			Z			Z	V				3			4	6	13
b) x 5-x = 4x	V				V		V							9	2	11
c) 6e+5f+e = 7e+5f	V		V		V		V							18		18
d) c-0 = c	V			L		L	V			7	1		1	6		15
e) 1bc-cc = 0	V	N	N		V	N	V	N			7			12	1	20
f) 4a-a = 3a					V		V				1			10	1	12
Fehlersumme 3	4	2	3	1	4	3	6	1	-	7	12	-	1	59	10	89
4 a) -ab-8ab = -9ab	V	Z			N	Z	V			2	2			5	8	17
b) -b+1b = 0	V	Z			N	N	V	N			7	2		6	3	18
c) -3cd-cd = cd	V	N			N	V	V	N			8			9	2	19
d) -5+e+1 = e-4	V	V	V		V	Z	V	V						20	1	21
e) -yx-xy = -2xy	V	R	N		N	O	V	N		2	5	[illegible]		5	9	22
f) -(5y-3y)+y = -y	V	K	Z	K	V	Z	V		5		5			6	5	21
Fehlersumme 4	6	6	3	1	6	6	6	4	5	4	27	3	-	51	26	116
Λ	21	13	12	5	19	19	22	9	6	13	60	3	11	190	78	361

	1	2	3	4	5	6	7	8	K	L	N	O	R	V	Z	
K (Klammerfehler)		1		1					6							
L (Lücke Auslassung)	1			1		1		1		13						
N (Nullenfehler)		5	4		6	3		6			60					
O (Falsche Operation)						1						3				
R (Rechenfehler) und Sonst.		1			3								11			
V (Variablenfehler)	19	2	4		9	2	22	2						190		
Z (Vorzeichenfehler)	1	4	4	3	1	8									78	

Aus einer solchen Klassenliste kann der Lehrer einmal insgesamt etwas über den bisherigen Erfolg des Unterrichts und den aktuellen Wissensstand der Klasse erfahren. Er sieht z.B., dass in seiner Klasse von $22 \cdot 24 = 528$ möglichen insgesamt 361 Aufgaben falsch gemacht wurden und dass die Hauptprobleme in seiner Klasse das Verständnis der Variablen (190 Variablenfehler), der Umgang mit der Null (60 Nullenfehler) sowie das Rechnen mit negativen Zahlen (78 Z-Fehler) sind.

Im Hinblick auf einzelne Aufgaben sieht er z.B., dass Aufgabe 2.c), 3.c) 4.d), e) und f) von mehr als 90 % der Schüler falsch gelöst wurden. Er kann daraus entnehmen, dass er mit der ganzen Klasse nochmals von Grund auf den Umgang mit Variablen und das Rechnen mit Null klären sollte.

Im Hinblick auf einzelne Schüler (untere Zeilen) erkennt er, dass Schülerin 1 und 7 überhaupt nicht mehr wissen, wie mit Variablen zu rechnen ist, dass bei Schüler 2 und 8 die Null das Hauptproblem darstellt, während Schüler 6 vor allem mit negativen Zahlen nicht zurechtkommt. Dementsprechend kann er Schülerin 1 und 7 bzw. Schüler 2 und 8 je in einer Kleingruppe zusammenfassen, wo er oder z.B. Schülerin 4 nochmals gezielt erklären und mit ihnen üben können.

Für die Therapie

Als Vorübung für das Rechnen mit Variablen ist es oft günstig, die Schüler Umformungen mit reinen Zahlentermen (Produkten) durchführen zu lassen, wobei ein Faktor nicht ausgerechnet, sondern in allen Umformungen beibehalten werden soll. Beispiel: $3 \cdot 10+8 \cdot 10+4 = 11 \cdot 10+4$. Anschliessend kann dieser Zahlenfaktor dann durch eine Variable ersetzt werden. Besonders dafür geeignete Zahlenfaktoren sind Zehnerpotenzen, da hier einstellige Koeffizienten auch nach Addition in der Zahl noch sichtbar bleiben.

Im Hinblick auf die einzelnen Fehlertypen sind einige Abhilfemöglichkeiten bei bestimmten Fehlertypen in der folgenden Tabelle (Tabelle 5) zusammengestellt.

TABELLE 5. Addition und Subtraktion von Termen: Fehler und Abhilfemöglichkeiten

Fehlertyp	Kurzerklärung	Handlungsanweisung	Aufgabentypen
V Variablen-fehler bei allen Aufgabentypen: z.B. 5a+2a=7aa	Buchstabe steht für Zahl: a ist z.B. eine Zahl, die sich Andrea gedacht hat		Vorübung mit Zahlentermen: Vereinfachen aber so, dass die unterstrichene Zahl stehen bleibt: Bsp 5·10 + 2·10 = dann 5·ⓐ + 2·ⓐ = dann 5·a + 2·a = (anschliessend verschiedene Zahlen einsetzen)
V Variablen-fehler nur bei bestimmten Aufgabentypen: z.B. 3e+4=7e 5a-a=5		fehlende Malpunkte schreiben	Im Anschluss jeweils ähnliche Aufgaben stellen, bei der vorher falsches Ergebnis richtig wird; Bsp. 3e + 4 = 3e + 4e = 5a - a = 5+a- a = und beide vergleichen
L Schuler betrachtet Produktterme wie "Wörter" z.B. xy + yx	yx=y·x, also kann man vertauschen	Malpunkte setzen, Produkte ordnen (erst Vorzeichen, dann Zahl, dann Buchstaben alphabetisch ordnen)	Sortieren lassen, welche Terme gleichartig sind: Bsp. -y·2·x x(-2)y x-2y...
O Falsche Operation, z.B. bei Klammern Multiplikation statt Addition $-3x+(2x+x)=$ $-6x^2-3x^2$		fehlende Malpunkte schreiben; bei Klammern immer zuerst untersuchen, welches Operationszeichen davor und welches dahinter steht	Aufgaben mit mal und plus oder minus vor oder hinter der Klammer vergleichen: Bsp. -2(x-3) 2-(x-3) (x-3)-2 (x-3)·(-2)
N Nullenfehler 0a+9a=10a bc+3bc=3bc 1bc-1bc=0bc (nicht zuende gerechnet)		fehlende Koeffizienten ("1") dazu schreiben	Im Anschluss an Aufgabe mit Null jeweils entsprechende mit Eins formulieren: Bsp. 0a+9a= 1a+9a= auch Aufgaben wie 0·b=, b·c·0 = stellen

(folgt)

TABELLE 5. (folgt)

Fehlertyp	Kurzerklärung	Handlungsanweisung	Aufgabentypen
K Klammerfehler: Bsp. -(x+3)=x-3 (Vorzeichen wird nur da, wo es steht, umgedreht)		Produkte ordnen; falls möglich, erst innerhalb der Klammer rechnen (bei 1. Term in Klammer evt. fehlendes Vorzeichen ergänzen), dann Klammern auflösen, dann restliche Operationen	Beispiele mit Zahlen auf verschiedene Arten rechnen: 6·10 + 3·10 = 9·10 oder = 60+30 ; -(10-3) = -7 oder = -10+3
Z Verwechslung mit Multiplikationsregel -x-2x=3x oder Vorzeichen bei Ergebnis vergessen oder falsche Operation: z.B. -2x+3x=-5x	Bei Addition oder Subtraktion von zwei Termen: Ergebnis hat gleiches Vorzeichen wie grösserer Betrag, da Add./Subtr. des kleineren Betrags den grösseren nicht uber/unter Null bringen kann	erst Klammern auflösen, dann Vorzeichen des Ergebnisses bestimmen (Vorzeichen des grösseren Betrags, dann Operation bestimmen (addieren bei gleichen, subtrahieren bei verschiedenen Zeichen)	zunächst reine Zahlenaufgaben, in denen alle Fälle vorkommen, dann parallel dazu entsprechende Aufgaben mit Variablen: Bsp. -2 +3 = -2x+3x=

SCHLUSS

Ein wichtiges Ergebnis dieser Untersuchung ist, dass sich bei der Addition und Subtraktion von Termen mehrere Schwierigkeitsfaktoren isolieren und in ihrem Gewicht abschätzen lassen.

Dabei handelt es sich in erster Linie um das Rechnen mit negativen Zahlen, das Vorkommen von Klammern und das Rechnen mit Null; ausserdem um Verschiedenartigkeit der Terme innerhalb einer Summe und verschiedene Reihenfolge der Variablen innerhalb eines Produkts.

Dem Lehrer steht damit ein Hilfsmittel zur Verfügung, mit dem er spezielle Lücken und Fehler sowohl in der ganzen Klasse als auch bei einzelnen Schülern diagnostizieren kann und das ihm Anhaltspunkte fur einen gezielten Abbau dieser Probleme gibt.

Er hat darüber hinaus ein Hilfsmittel zur Konstruktion von Aufgaben zur Hand, das es ihm erlaubt, die Schwierigkeiten von Aufgaben zu dosieren und auch Aufgaben mit vorgegebenen Schwierigkeitskombinationen in beliebiger Anzahl vom Computer stellen zu lassen.

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